Lecture #7 Vector Differentiation and Integration

I. Differential Vector Operators

1. Vector fields $\underset{\sim}{V}(x,y,z)$ and scalar fields $\phi(x,y,z)$ may be differentiated with respect to spatial dimensions

A. Gradient, ∇

1. Characterizes the change of a scalar quantity with position.
2. In $\mathbb{R}^3$, label coordinates x_1, x_2, & x_3

Scalar field $\longrightarrow$ $\phi(\underset{\sim}{r})$ at $\underset{\sim}{r} = x_1\hat{\underset{\sim}{e}}_1 + x_2\hat{\underset{\sim}{e}}_2 + x_3\hat{\underset{\sim}{e}}_3$

3. $d\phi = \left(\frac{\partial\phi}{\partial x_1}\right)dx_1 + \left(\frac{\partial\phi}{\partial x_2}\right)dx_2 + \left(\frac{\partial\phi}{\partial x_3}\right)dx_3$

a. This can be written $\nabla\phi \cdot d\underset{\sim}{r}$ where

$$\nabla\phi = \begin{pmatrix} \partial\phi/\partial x_1 \\ \partial\phi/\partial x_2 \\ \partial\phi/\partial x_3 \end{pmatrix} \qquad d\underset{\sim}{r} = \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$ (column vector notation)

or

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x_1}\right)\hat{\underset{\sim}{e}}_1 + \left(\frac{\partial\phi}{\partial x_2}\right)\hat{\underset{\sim}{e}}_2 + \left(\frac{\partial\phi}{\partial x_3}\right)\hat{\underset{\sim}{e}}_3$$

$$d\underset{\sim}{r} = dx_1\hat{\underset{\sim}{e}}_1 + dx_2\hat{\underset{\sim}{e}}_2 + dx_3\hat{\underset{\sim}{e}}_3$$

4. NOTE: a. $\nabla\phi$ is a vector (no under tilde)

b. It can be shown to transform under a rotation $\underset{\sim}{S}$, $(\nabla\phi)' = \underset{\sim}{S}(\nabla\phi)$

5. Vector Differential Operator: $\boxed{\nabla = \hat{\underset{\sim}{e}}_x\frac{\partial}{\partial x} + \hat{\underset{\sim}{e}}_y\frac{\partial}{\partial y} + \hat{\underset{\sim}{e}}_z\frac{\partial}{\partial z}}$

a. Operates on a scalar to produce a vector.

b. Operates only on what is to the right $\Rightarrow$ order matters! $\nabla\phi$ (Not $\phi\nabla$).

6. Ex: Force expressed as gradient of scalar potential $V(\underset{\sim}{r})$

$$\boxed{\underset{\sim}{F} = -\nabla V(\underset{\sim}{r})}$$

7. Ex: $\nabla r = \frac{\partial r}{\partial x}\hat{e}_x + \frac{\partial r}{\partial y}\hat{e}_y + \frac{\partial r}{\partial z}\hat{e}_z$

a. $\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{\frac{1}{2}} = \frac{\frac{1}{2}2x}{(x^2+y^2+z^2)^{\frac{1}{2}}} = \frac{x}{r}$, Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

b. Thus

$$\nabla r = \frac{x}{r}\hat{e}_x + \frac{y}{r}\hat{e}_y + \frac{z}{r}\hat{e}_z = \frac{1}{r}\underbrace{(x\hat{e}_x + y\hat{e}_y + z\hat{e}_z)}_{=\mathbf{r}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}$$

c. So $\boxed{\nabla r = \hat{\mathbf{r}}}$ Also useful is $\boxed{\hat{\mathbf{r}} = \frac{x}{r}\hat{e}_x + \frac{y}{r}\hat{e}_y + \frac{z}{r}\hat{e}_z}$

8. Ex: Central Force $V(\mathbf{r}) = \frac{1}{r}$ ← spherically symmetric potential

a. $\mathbf{F} = -\nabla V(\mathbf{r}) = -\nabla(\frac{1}{r})$

b. $\frac{\partial}{\partial x}(\frac{1}{r}) = \frac{-1}{r^2}\frac{\partial r}{\partial x}$

c. Thus

$$\mathbf{F} = -\nabla(\tfrac{1}{r}) = +\frac{1}{r^2}\frac{\partial r}{\partial x}\hat{e}_x + \frac{1}{r^2}\frac{\partial r}{\partial y}\hat{e}_y + \frac{1}{r^2}\frac{\partial r}{\partial z}\hat{e}_z = +\frac{1}{r^2}\nabla r = \frac{+1}{r^2}\hat{\mathbf{r}}$$

d. So $\boxed{\mathbf{F} = \frac{1}{r^2}\hat{\mathbf{r}}}$ ⇒ Spherical potential yields radial force!

B. Divergence, $\nabla \cdot$

1. Def: $\boxed{\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$ (Dot Product of ∇ and $\mathbf{A}$)

2. Ex: $\nabla \cdot \mathbf{r} = \left(\hat{e}_x\frac{\partial}{\partial x} + \hat{e}_y\frac{\partial}{\partial y} + \hat{e}_z\frac{\partial}{\partial z}\right)\cdot(x\hat{e}_x + y\hat{e}_y + z\hat{e}_z)$

a. NOTE: Cartesian unit vectors are constant, so $\frac{\partial}{\partial x_i}\hat{e}_i = 0$!

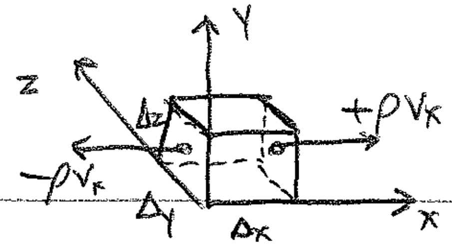
b. $= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = \boxed{3}$

3. Physical Significance of Divergence:

a. Consider the flow $\mathbf{v}(\mathbf{r})$ of a fluid of mass density $\rho(\mathbf{r})$.

⇒ Mass flow rate at any point $\mathbf{r}$ is $\boxed{\rho(\mathbf{r})\mathbf{v}(\mathbf{r})}$

b. Consider rate of change of mass $\frac{\Delta m}{\Delta t}$ due to flow along v_x



$$\frac{\Delta m}{\Delta t} = -\rho v_x\Big|_{0,\frac{\Delta y}{2},\frac{\Delta z}{2}} \Delta y \Delta z + \rho v_x\Big|_{\Delta x,\frac{\Delta y}{2},\frac{\Delta z}{2}} \Delta y \Delta z$$

$$= \left(\frac{\rho v_x\Big|_{\Delta x,\frac{\Delta y}{2},\frac{\Delta z}{2}} - \rho v_x\Big|_{0,\frac{\Delta y}{2},\frac{\Delta z}{2}}}{\Delta x}\right) \Delta x \Delta y \Delta z$$

c. **NOTE:** $$\lim_{\Delta x \to 0} \frac{f(\Delta x, y, z) - f(0, y, z)}{\Delta x} = \frac{\partial f}{\partial x}$$

d. But, we can also have flow in or out along $\hat{\underset{\sim}{e}}_y$ or $\hat{\underset{\sim}{e}}_z$!

$$\frac{\Delta m}{\Delta t} = \left[\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right] \Delta x \Delta y \Delta z = \nabla \cdot (\rho \underset{\sim}{v}) \Delta x \Delta y \Delta z$$

e. So $$\frac{\left(\frac{\Delta m}{V}\right)}{\Delta t} = \nabla \cdot (\rho \underset{\sim}{v}) \qquad \frac{m}{V} = \rho \qquad \Rightarrow \boxed{\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho \underset{\sim}{v})}$$ Equation of Continuity

Volume $V = \Delta x \Delta y \Delta z$

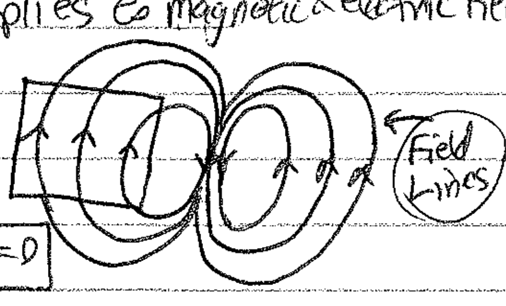
f. "Divergence" describes net outflow from a small volume element $\Rightarrow$ reduces mass density.

4. Zero Divergence of a Vector Field

a. Zero Divergence means a steady-state flux within region. "What goes in, comes out"

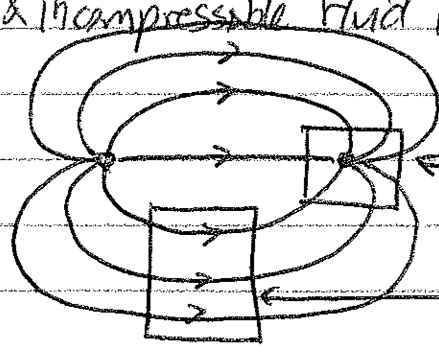
b. Applies to magnetic & electric fields & incompressible fluid flow.

"Dipole Field"



Flux in = Flux out $\Rightarrow \boxed{\nabla \cdot \underset{\sim}{B} = 0}$

"Sink"



$\boxed{\nabla \cdot \underset{\sim}{E} \neq 0}$

$\boxed{\nabla \cdot \underset{\sim}{E} = 0}$

5. <u>Terminology</u>. $\nabla\cdot\underset{\sim}{B}=0$ everywhere $\Rightarrow$ $\underset{\sim}{B}$ is <u>solenoidal</u>.

b. No sources or sinks (No magnetic monopoles).

C. <u>Curl</u>, $\nabla\times$

1. <u>Def:</u>

$$\boxed{\nabla\times\underset{\sim}{V}=\underset{\sim}{\hat{e}}_x\left(\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right)+\underset{\sim}{\hat{e}}_y\left(\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right)+\underset{\sim}{\hat{e}}_y\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)=\begin{vmatrix}\underset{\sim}{\hat{e}}_x & \underset{\sim}{\hat{e}}_y & \underset{\sim}{\hat{e}}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ V_x & V_y & V_z\end{vmatrix}}$$

(Cross Product of ∇ and $\underset{\sim}{V}$)

Top down evaluation "order matters!"

2. <u>Ex: Curl of a central force field</u>: $\underset{\sim}{F}=f(r)\,\underset{\sim}{\hat{r}}$

a. $\nabla\times\left[f(r)\,\underset{\sim}{\hat{r}}\right]$ where $\underset{\sim}{\hat{r}}=\frac{x}{r}\underset{\sim}{\hat{e}}_x+\frac{y}{r}\underset{\sim}{\hat{e}}_y+\frac{z}{r}\underset{\sim}{\hat{e}}_z$

b. <u>x-component</u>: $\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}=\frac{\partial}{\partial y}\left(f(r)\frac{z}{r}\right)-\frac{\partial}{\partial z}\left(f(r)\frac{y}{r}\right)$

Since $\frac{\partial z}{\partial y}=0$

$$=z\frac{\partial}{\partial y}\left(\frac{f(r)}{r}\right)-y\frac{\partial}{\partial z}\left(\frac{f(r)}{r}\right)=z\frac{\partial r}{\partial y}\frac{\partial}{\partial r}\left(\frac{f(r)}{r}\right)-y\frac{\partial r}{\partial z}\frac{\partial}{\partial r}\left(\frac{f(r)}{r}\right)$$

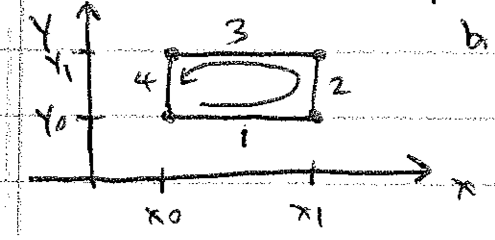
$\frac{\partial}{\partial y}=\frac{\partial}{\partial r}\frac{\partial r}{\partial y}$, etc.

c. Finally, note $\frac{\partial r}{\partial y}=\frac{y}{r}$, $\frac{\partial r}{\partial z}=\frac{z}{r}$, so $=\frac{zy}{r}\frac{\partial}{\partial r}\left(\frac{f(r)}{r}\right)-\frac{yz}{r}\frac{\partial}{\partial r}\left(\frac{f(r)}{r}\right)$

$=0$!

d. All other components yield $0\Rightarrow\boxed{\nabla\times\left[f(r)\,\underset{\sim}{\hat{r}}\right]=0}$

3. Geometric Interpretation:

a. 

b. $\oint\underset{\sim}{B}\cdot d\underset{\sim}{S}=\int_{x_0}^{x_1}B_x\,dx+\int_{y_0}^{y_1}B_y\,dy-\int_{x_1}^{x_0}B_x\,dx$

c. If contributions do not cancel, $-\int_{y_1}^{y_0}B_y\,dy$.
the $\nabla\times\underset{\sim}{B}=0$.

I. C. (Continued)

4. Terminology: a. Circulation $= \dfrac{\oint \underset{\sim}{B} \cdot d\underset{\sim}{s}}{\oint d\underset{\sim}{s}}$ (per unit area)

$= (\nabla \times \underset{\sim}{B}) \cdot \hat{\underset{\sim}{n}} \leftarrow$ Normal to loop $d\underset{\sim}{s}$.

b. Fluid dynamics: Vorticity $\underset{\sim}{\omega} \equiv \nabla \times \underset{\sim}{V}$

c. $\nabla \times \underset{\sim}{B} = 0$ everywhere $\Rightarrow$ $\underset{\sim}{B}$ is irrotational.

D. Higher Order Derivatives:

1. Laplacian: $\boxed{\nabla^2 \equiv \nabla \cdot \nabla = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}}$

2. Irrotational and Solenoidal Vector Fields:

a. $\boxed{\nabla \times \nabla \phi = 0}$ Any gradient has a vanishing curl $\Rightarrow$ irrotational!

b. $\boxed{\nabla \cdot (\nabla \times \underset{\sim}{V}) = 0}$ Any curl has a vanishing divergence $\Rightarrow$ Solenoidal!

3. Vector Laplacian, $\nabla^2 \underset{\sim}{V}$ (Laplacian of a vector).

a. $\boxed{\nabla \times (\nabla \times \underset{\sim}{V}) = \nabla(\nabla \cdot \underset{\sim}{V}) - \nabla^2 \underset{\sim}{V}}$

4. Ex: Maxwell's Equations (SI Units)

Sources: Charge density $\rho(\underset{\sim}{r})$
Current density $\underset{\sim}{J}(\underset{\sim}{r})$

a. ① $\nabla \cdot \underset{\sim}{B} = 0$

② $\nabla \cdot \underset{\sim}{E} = \rho / \varepsilon_0$

③ $\nabla \times \underset{\sim}{B} = \varepsilon_0 \mu_0 \dfrac{\partial \underset{\sim}{E}}{\partial t} + \mu_0 \underset{\sim}{J}$

④ $\nabla \times \underset{\sim}{E} = -\dfrac{\partial \underset{\sim}{B}}{\partial t}$

where $\dfrac{1}{c^2} = \mu_0 \varepsilon_0$

μ_0: magnetic permeability; ε_0: electric permittivity

I. D.4 (Continued)

b. Electromagnetic Waves in Vaccuum $(\rho=0,\ \underset{\sim}{J}=0)$

i. $\frac{\partial}{\partial t}$ (3) $\Rightarrow \frac{\partial}{\partial t}(\nabla\times\underset{\sim}{B}) = \varepsilon_0\mu_0\frac{\partial^2 \underset{\sim}{E}}{\partial t^2}$

ii. $\nabla\times$ (4) $\Rightarrow \nabla\times(\nabla\times\underset{\sim}{E}) = -\frac{\partial}{\partial t}(\nabla\times\underset{\sim}{B}) = -\varepsilon_0\mu_0\frac{\partial^2 \underset{\sim}{E}}{\partial t^2}$

iii. Use $\nabla\times(\nabla\times\underset{\sim}{E}) = \nabla(\underbrace{\nabla\cdot\underset{\sim}{E}}_{\nabla\cdot\underset{\sim}{E}=0 \text{ for } \rho=0!}) - \nabla^2\underset{\sim}{E}$ and $\varepsilon_0\mu_0 = \frac{1}{c^2}$

iv. $$\boxed{\nabla^2\underset{\sim}{E} = \frac{1}{c^2}\frac{\partial^2\underset{\sim}{E}}{\partial t^2}}$$ Wave equation for propagation of light waves at speed c.

5. Other Vector Identities:

a. $\nabla\cdot(f\underset{\sim}{V}) = (\nabla f)\cdot\underset{\sim}{V} + f\,\nabla\cdot\underset{\sim}{V}$

b. $\nabla\times(f\underset{\sim}{V}) = f(\nabla\times\underset{\sim}{V}) + (\nabla f)\times\underset{\sim}{V}$

c. $\nabla(\underset{\sim}{A}\cdot\underset{\sim}{B}) = (\underset{\sim}{B}\cdot\nabla)\underset{\sim}{A} + (\underset{\sim}{A}\cdot\nabla)\underset{\sim}{B} + \underset{\sim}{B}\times(\nabla\times\underset{\sim}{A}) + \underset{\sim}{A}\times(\nabla\times\underset{\sim}{B})$

II. Vector Integration

A. Line Integrals

1. Forms: $\int_C \phi\, d\underset{\sim}{r}$ $\quad \int_C \underset{\sim}{E}\cdot d\underset{\sim}{r}$ $\quad \int_C \underset{\sim}{V}\times d\underset{\sim}{r}$

2. Using $d\underset{\sim}{r} = dx\,\hat{\underset{\sim}{e}}_x + dy\,\hat{\underset{\sim}{e}}_y + dz\,\hat{\underset{\sim}{e}}_z$, [noting $\phi = \phi(x,y,z)$]

NOTE: answer is a vector! $\longrightarrow$ $$\int_C \phi\, d\underset{\sim}{r} = \hat{\underset{\sim}{e}}_x\int_C \phi\, dx + \hat{\underset{\sim}{e}}_y\int_C \phi\, dy + \hat{\underset{\sim}{e}}_z\int \phi\, dz$$ $\leftarrow$ Sum of Scalar integrals!

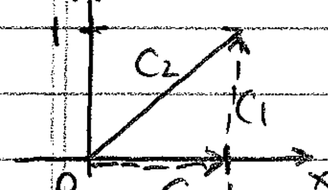
3. NOTE! The path must be specified!

a. $\int_C \phi(x,y,z)\,dx$ where we need to know $y(x)$ and $z(x)$. Equation for path.

4. Work: $W = \int_C \underset{\sim}{F} \cdot d\underset{\sim}{r}$

NOTE: Answer is a scalar! $\longrightarrow = \int_C F_x\,dx + \int_C F_y\,dy + \int_C F_z\,dz$

5. Path Dependence: a. $\underset{\sim}{F} = -y\,\hat{\underset{\sim}{e}}_x + x\,\hat{\underset{\sim}{e}}_y$



b. $\int_{C_1} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^1 dx\, F_x(x,0) + \int_0^1 dy\, F_y(1,y)$

$= \cancel{\int_0^1 -(0)\,dx} + \int_0^1 (1)\,dy = \boxed{1}$

$C_2 \Rightarrow x = y$

c. $\int_{C_2} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^1 F_x(x,x)\,dx + \int_0^1 F_y(y,y)\,dy = \int_0^1 (-x)\,dx + \int_0^1 y\,dy = -\frac{1}{2} + \frac{1}{2} = \boxed{0}$

$\boxed{\text{Value of line integral depends on path!}}$

Unless integrand has special properties, line integrals depend on path!

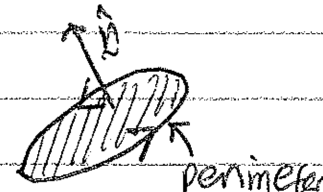
B. Surface Integrals

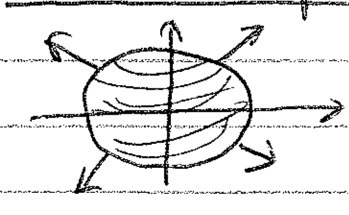
1. Forms: $\int \phi\, d\underset{\sim}{\sigma}$ $\quad \int \underset{\sim}{V} \cdot d\underset{\sim}{\sigma}$ $\quad \int \underset{\sim}{V} \times d\underset{\sim}{\sigma}$

2. a. Often $d\underset{\sim}{\sigma} = \hat{\underset{\sim}{n}}\,dA$, where $\hat{\underset{\sim}{n}}$ is normal to area dA.

b. Which direction normal? i. Closed Surface: Outward normal is positive!

ii. Open Surface: Right-hand Rule





3. <u>Most Common Form:</u> $\int \underset{\sim}{V} \cdot d\underset{\sim}{\sigma}$ ← Flow or flux through a surface.

C. <u>Volume Integrals:</u>

1.

$$\boxed{\int \underset{\sim}{V}\, d\tau = \underset{\sim}{\hat{e}}_x \int V_x\, d\tau + \underset{\sim}{\hat{e}}_y \int V_y\, d\tau + \underset{\sim}{\hat{e}}_z \int V_z\, d\tau}$$

(Vector sum of scalar integrals)

2. For terms which vanish at infinity,

$$\int f(r)\, \nabla \cdot \underset{\sim}{A}(\underset{\sim}{r})\, d\tau = -\int \underset{\sim}{A}(\underset{\sim}{r}) \cdot \nabla f(r)\, d\tau$$

$$\int \underset{\sim}{C}(\underset{\sim}{r}) \cdot [\nabla \times \underset{\sim}{A}(\underset{\sim}{r})]\, d\tau = \int \underset{\sim}{A}(\underset{\sim}{r}) \cdot [\nabla \times \underset{\sim}{C}(\underset{\sim}{r})]\, d\tau$$

Can be proven using integration by parts