Lecture #8 Integral Theorems and Potential Theory

I. Integral Theorems

A. Gauss' Theorem

1. Theorem: For a vector $\underset{\sim}{A}$ with continuous first derivatives over a simply connected volume in $\mathbb{R}^3$,

$$\boxed{\oint_{\partial V} \underset{\sim}{A} \cdot d\underset{\sim}{\sigma} = \int_V \nabla \cdot \underset{\sim}{A} \, d\tau}$$

Closed surface enclosing volume

Volume

2. Relates the divergence of a vector throughout a volume to the flux of that vector through the surface.

3. If the volume is all $\mathbb{R}^3$ (boundaries at infinity) and the volume integral converges, then $\int_V \nabla \cdot \underset{\sim}{A} \, d\tau = 0$

B. Green's Theorem

1. Theorem: For two scalar functions, u & v,

$$\int_V (u\nabla^2 v - v\nabla^2 u)\, d\tau = \oint_{\partial V} (u\nabla v - v\nabla u) \cdot d\underset{\sim}{\sigma}$$

2. Proof: a. $\nabla \cdot (u\nabla v) = u\nabla^2 v + \nabla u \cdot \nabla v$

$$\nabla \cdot (f \underset{\sim}{V}) = f \nabla \cdot \underset{\sim}{V} + (\nabla f) \cdot \underset{\sim}{V}$$

$f = u \quad \underset{\sim}{V} = \nabla v$

b. Similarly $\nabla \cdot (v\nabla u) = v\nabla^2 u + \nabla v \cdot \nabla u$

c. Thus $\int_V (u\nabla^2 v - v\nabla^2 u)\, d\tau = \int_V [\nabla \cdot (u\nabla v) - \cancel{\nabla u \cdot \nabla v} + \cancel{\nabla v \cdot \nabla u} - \nabla \cdot (v\nabla u)]\, d\tau$

d. Thus $\int_V \nabla\cdot(U\nabla V - V\nabla U)\,d\tau \overset{\text{By Gauss' Theorem}}{=} \oint_{\partial V}(U\nabla V - V\nabla U)\cdot d\underset{\sim}{\sigma}$. ✓

3. Alternate form: $\boxed{\oint_{\partial V} U\nabla V\cdot d\underset{\sim}{\sigma} = \int_V U\nabla^2 V\,d\tau + \int_V \nabla U\cdot\nabla V\,d\tau}$

C. Gauss' Theorem for Gradient and Curl

1. Gradient Version: Scalar function $B(x,y,z)$

$$\boxed{\oint_{\partial V} B\,d\underset{\sim}{\sigma} = \int_V \nabla B\,d\tau}$$

2. Curl Version: Vector $\underset{\sim}{P}$

$$\boxed{\oint_{\partial V} d\underset{\sim}{\sigma}\times\underset{\sim}{P} = \int_V (\nabla\times\underset{\sim}{P})\,d\tau}$$

3. Useful strategy for proving alternate theorems:

a. Take $\underset{\sim}{B}(x,y,z) = B(x,y,z)\,\underset{\sim}{a}$

where $\underset{\sim}{a}$ is an arbitrary vector of constant magnitude and direction.

b. $\nabla\cdot(B\underset{\sim}{a}) = \nabla B\cdot\underset{\sim}{a} + B\,\cancel{\nabla\cdot\underset{\sim}{a}}^{\,0}$

c. $\int_V \nabla\cdot(B\underset{\sim}{a})\,d\tau = \int_V \nabla B\cdot\underset{\sim}{a}\,d\tau$

$\downarrow$ Gauss' Thm $\qquad \downarrow$ $\underset{\sim}{a}$ is constant

d. $\oint_{\partial V} B\underset{\sim}{a}\cdot d\underset{\sim}{\sigma} = \underset{\sim}{a}\cdot\int_V \nabla B\,d\tau$

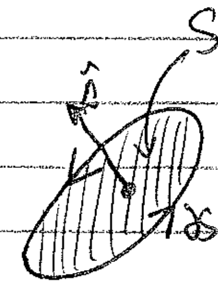
$\downarrow$ $\underset{\sim}{a}$ constant

e. $\underset{\sim}{a}\cdot\oint_{\partial V} B\,d\underset{\sim}{\sigma} = \underset{\sim}{a}\cdot\int_V \nabla B\,d\tau \Rightarrow \underset{\sim}{a}\cdot\left[\oint_{\partial V} B\,d\underset{\sim}{\sigma} - \int_V \nabla B\,d\tau\right] = 0$

f. Since $\underset{\sim}{a}$ is arbitrary, quantity in brackets must be zero! $\Rightarrow \boxed{\oint_{\partial V} B\,d\underset{\sim}{\sigma} = \int_V \nabla B\,d\tau}$

D. Stoke's Theorem

1. Theorem:
$$\boxed{\oint_{\partial S} \mathbf{B}\cdot d\mathbf{r} = \int_S \nabla\times\mathbf{B}\cdot d\boldsymbol{\sigma}}$$



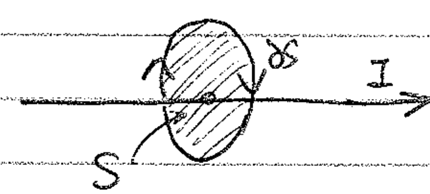
2. Relates the curl of a vector over a surface to the line integral around the perimeter
   a. NOTE: Sign of line integral changes when integrating opposite direction, but so does normal for $d\boldsymbol{\sigma} = \hat{\mathbf{n}}\,dA$.

3. For a closed surface, there is no perimeter along which to integrate, so
$$\boxed{\oint_S \nabla\times\mathbf{B}\cdot d\boldsymbol{\sigma} = 0}$$
   a. Note:
$$\oint_{S=\partial V} \nabla\times\mathbf{B}\cdot d\boldsymbol{\sigma} \underset{\text{Gauss' Thm}}{=} \int_V \underbrace{\nabla\cdot(\nabla\times\mathbf{B})}_{=0\,!}\,d\tau = 0\,!$$

4. Gradient Version:
$$\boxed{\int_S d\boldsymbol{\sigma}\times\nabla\phi = \oint_{\partial S}\phi\, d\mathbf{r}}$$

5. Curl Version:
$$\boxed{\int_S (d\boldsymbol{\sigma}\times\nabla)\times\mathbf{P} = \oint_{\partial S} d\mathbf{r}\times\mathbf{P}}$$

6. Ex: Oersted's Law



   a. Consider a wire carrying a time-independent current I
   b. $\nabla\times\mathbf{B} = \cancel{\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}}^{\,0} + \mu_0\mathbf{J}$

$$I = \int_S \mathbf{J}\cdot d\boldsymbol{\sigma} = \frac{1}{\mu_0}\int_S (\nabla\times\mathbf{B})\cdot d\boldsymbol{\sigma} \overset{\text{Stoke's Thm.}}{=} \frac{1}{\mu_0}\oint_{\partial S}\mathbf{B}\cdot d\mathbf{r} \Rightarrow \overset{\text{Oersted's Law.}}{\boxed{I = \frac{1}{\mu_0}\oint_{\partial S}\mathbf{B}\cdot d\mathbf{r}}}$$

II. Potential Theory

A. Scalar Potential

1. Some forces may be expressed as the gradient of a scalar potential ϕ

$$\boxed{\underset{\sim}{F} = -\nabla\phi}$$

a. A force has 3 components $\underset{\sim}{F}(x,y,z)$, whereas a potential has one $\phi(x,y,z)$
$\Rightarrow$ more simple mathematical description

b. Potential determined only to an additive constant $\Rightarrow$ only differences matter!

2. What conditions must $\underset{\sim}{F}$ satisfy for ϕ to exist?

a. Work: $W = \int_A^B \underset{\sim}{F} \cdot d\underset{\sim}{r} = -\int_A^B \nabla\phi \cdot d\underset{\sim}{r} = -\int_A^B d\phi = \phi(A) - \phi(B)$

i. $\nabla\phi \cdot d\underset{\sim}{r} = \left(\hat{\underset{\sim}{e}}_x \frac{\partial\phi}{\partial x} + \hat{\underset{\sim}{e}}_y \frac{\partial\phi}{\partial y} + \hat{\underset{\sim}{e}}_z \frac{\partial\phi}{\partial z}\right) \cdot \left(dx\, \hat{\underset{\sim}{e}}_x + dy\, \hat{\underset{\sim}{e}}_y + dz\, \hat{\underset{\sim}{e}}_z\right) = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$

ii. Thus, value of integral is <u>independent of the path</u>
$\Rightarrow$ depends only on end points.

iii. $\oint \underset{\sim}{F} \cdot d\underset{\sim}{r} = 0$ over closed path.

iv. <u>Conservative Force</u>: Work done is independent of path.

b. <u>Curl</u>: $\nabla \times \underset{\sim}{F} = \nabla \times (-\nabla\phi) = 0$!
<u>Force must have zero curl.</u>

3. <u>Ex</u>: Compute Gravitational Potential from Force Law

$$\underset{\sim}{F}_G = -\frac{G m_1 m_2}{r^2}\hat{\underset{\sim}{r}} = -\frac{k\hat{\underset{\sim}{r}}}{r^2}$$

a. If $\underset{\sim}{F}_G = -\nabla\phi_G$, we must integrate.

$$\int_\infty^r \underset{\sim}{F}_G \cdot d\underset{\sim}{r} = -\int_\infty^r \nabla\phi_G \cdot d\underset{\sim}{r} = -\int_\infty^r d\phi_G = \cancel{\phi_G(\infty)}^{\,0} - \phi_G(r)$$

$\leftarrow$ Choose reference $\phi_G(\infty) = 0$!

b. So $\phi_G(r) = -\int_\infty^r \underset{\sim}{F_G} \cdot d\underset{\sim}{r} = -\int_\infty^r \left(-\frac{k\hat{r}}{r^2}\right) \cdot d\underset{\sim}{r} = +\int_\infty^r \frac{k\,dr}{r^2}$

$$= \left.\frac{-k}{r}\right|_\infty^r = -\frac{k}{r} + \cancel{\frac{k}{\infty}}^{0} = \boxed{-\frac{Gm_1m_2}{r} = \phi_G(r)}$$

B. Vector Potential:

1. For a vector that is solenoidal (divergence free), we may enforce this property by introducing a vector potential.

$$\boxed{\underset{\sim}{B} = \nabla \times \underset{\sim}{A}}$$

a. $\nabla \cdot \underset{\sim}{B} = \nabla \cdot (\nabla \times \underset{\sim}{A}) = 0$! Very useful for magnetic field!

2a. In fact, if $\underset{\sim}{B}$ is solenoidal, a vector potential $\underset{\sim}{A}$ exists.

b. But, the vector potential is not unique!

c. You may add not only an arbitrary constant, but also the gradient of any scalar function $\nabla\phi$ without changing $\underset{\sim}{B}$.

$$\underset{\sim}{B} = \nabla \times (\underset{\sim}{A} + \nabla\phi) = \nabla \times \underset{\sim}{A} + \cancel{\nabla \times \nabla\phi}^{0} = \nabla \times \underset{\sim}{A}.$$

C. Electromagnetic Scalar and Vector Potentials

1. Choosing $\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$ to always satisfy $\nabla \cdot \underset{\sim}{B} = 0$, we can substitute into Faraday's Law,

$$\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t} = -\nabla \times \frac{\partial \underset{\sim}{A}}{\partial t} \Rightarrow \nabla \times \left[\underset{\sim}{E} + \frac{\partial \underset{\sim}{A}}{\partial t}\right] = 0$$

2. Since $\underset{\sim}{E} + \frac{\partial \underset{\sim}{A}}{\partial t}$ is curl-free, it may be written as the gradient of a scalar, $-\nabla\phi$.

3. Thus

$$\boxed{\underset{\sim}{B} = \nabla \times \underset{\sim}{A}} \qquad \boxed{\underset{\sim}{E} = -\nabla\phi - \frac{\partial \underset{\sim}{A}}{\partial t}}$$

But $\underset{\sim}{A}$ is still arbitrary because you may add a gradient of a scalar!

4. <u>Gauge Condition:</u> Choice to specify form of $\underset{\sim}{A}$.

Ex)
a. <u>Lorentz Gauge</u>: $\boxed{\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot \underset{\sim}{A} = 0}$

b. Coulomb's Law: $\frac{\rho}{\varepsilon_0} = \nabla \cdot \underset{\sim}{E} = -\nabla^2 \phi - \frac{\partial}{\partial t}\nabla \cdot \underset{\sim}{A} = -\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}$

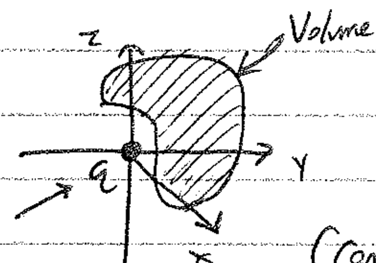
$\Rightarrow$ Lorentz gauge decouples $\underset{\sim}{A}$ & ϕ such that ϕ is entirely determined by ρ

D. <u>Gauss' Law</u> (not Gauss' Thm)

1. <u>Gauss' Law</u>:
$$\boxed{\oint_{\partial V} \underset{\sim}{E} \cdot d\sigma = \begin{cases} \frac{q}{\varepsilon_0} & \text{if } \partial V \text{ encloses charge } q \\ 0 & \text{if } \partial V \text{ does not enclose } q. \end{cases}}$$

2. <u>Proof</u>: a. Electric Field for a point charge q is $\underset{\sim}{E} = \frac{q\,\hat{\underset{\sim}{r}}}{4\pi\varepsilon_0 r^2}$

b.



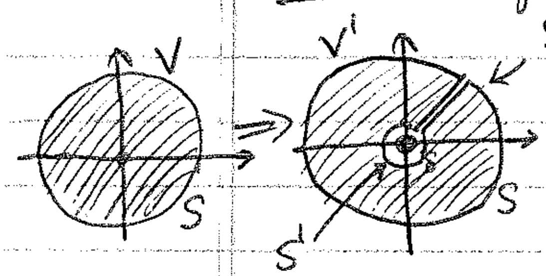
<u>Case 1:</u> Charge not in volume

$$\int_V \nabla \cdot \underset{\sim}{E}\, d\tau = \oint_{\partial V} \underset{\sim}{E} \cdot d\sigma = 0$$

But $\nabla \cdot \underset{\sim}{E} = 0$

(Central force has zero divergence)

c. <u>Case 2</u>: Charge in Volume



Simply connected region (not enclosing q.)

$$\int_{\partial V'} \underset{\sim}{E} \cdot d\sigma = \oint_S \underset{\sim}{E} \cdot d\sigma + \frac{q}{4\pi\varepsilon_0}\oint_{S'} \frac{\hat{\underset{\sim}{r}} \cdot d\sigma'}{\delta^2} = 0$$

δ: radius of inner surface S'

$d\sigma' = -\hat{\underset{\sim}{r}}\, dA = -\hat{\underset{\sim}{r}}\,(\delta^2 d\Omega)$

d. $$\oint \frac{\hat{\underset{\sim}{r}} \cdot d\sigma'}{\delta^2} = \oint \frac{\hat{\underset{\sim}{r}} \cdot (-\hat{\underset{\sim}{r}}\, \cancel{\delta^2}\, d\Omega)}{\cancel{\delta^2}} = -\oint d\Omega = -4\pi$$

e. Thus $\oint_S \underset{\sim}{E} \cdot d\sigma = \frac{+q}{\cancel{4\pi}\varepsilon_0}(+\cancel{4\pi}) = \frac{q}{\varepsilon_0}$ ✓

II. (Continued)

E. Poisson's Equation

1. For a time independent situation, $\underset{\sim}{E} = -\nabla\phi$

2. $\nabla \cdot \underset{\sim}{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \boxed{\nabla^2\phi = \frac{-\rho}{\varepsilon_0}}$ Poisson's Equation

3. If charge density $\rho = 0$, $\Rightarrow \boxed{\nabla^2\phi = 0}$ Laplace's Equation

4. Handling derivatives at $\underset{\sim}{r}=0$ for a point charge q at $\underset{\sim}{r}=0$:

a. Dirac Delta Function: $\boxed{\nabla^2\phi = -\frac{q}{\varepsilon_0}\delta(\underset{\sim}{r})}$ (charge q at $\underset{\sim}{r}=0$)

b. Potential for a point charge is $\phi = \frac{q}{4\pi\varepsilon_0}\frac{1}{r} \Rightarrow \frac{q}{4\pi\varepsilon_0}\nabla^2\left(\frac{1}{r}\right) = -\frac{q}{\varepsilon_0}\delta(\underset{\sim}{r})$

c. Thus, we obtain $\boxed{\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(\underset{\sim}{r})}$ Handles derivatives that don't exist at $\underset{\sim}{r}=0$.

d. Can also be written $\boxed{\nabla_1^2\left(\frac{1}{r_{12}}\right) = -4\pi\delta(\underset{\sim}{r_1}-\underset{\sim}{r_2})}$

where $r_{12} = |\underset{\sim}{r_1}-\underset{\sim}{r_2}|$ and ∇_1 implies derivatives apply to $\underset{\sim}{r_1}$.

F. Helmholtz's Theorem

1. Two theorems establish conditions for uniqueness and existence of solutions in time-independent electromagnetic problems.

2. Theorem One:

> A vector field is uniquely specified by giving its divergence and curl within a simply connected region and its normal component on the boundary.

Still applies even when delta functions are needed within the region.

3. <u>Restated:</u> $\mathbf{P}$ is unique if $\nabla\cdot\mathbf{P}=s$ and $\nabla\times\mathbf{P}=\mathbf{c}$ are given in volume and P_n is specified on boundary.

4. <u>Helmholtz's Theorem:</u>
A vector $\mathbf{P}$ with both source and circulation densities vanishing at infinity may be written as the sum of two parts, one solenoidal and one irrotational.

$$\boxed{\mathbf{P} = -\nabla\phi + \nabla\times\mathbf{A}}$$

5. $\Rightarrow$ General form for any well-behaved vector field.

6. Helmholtz's Theorem legitimizes the division of quantities in electromagnetic theory into irrotational $\mathbf{E}$ and solenoidal $\mathbf{B}$

a. $s = \nabla\cdot\mathbf{P}$ $\Rightarrow$ charge density

b. $\mathbf{c} = \nabla\times\mathbf{P}$ $\Rightarrow$ current density