

# Lecture #9 Curvilinear Coordinates

## I. Curvilinear Coordinates

### A. General Properties

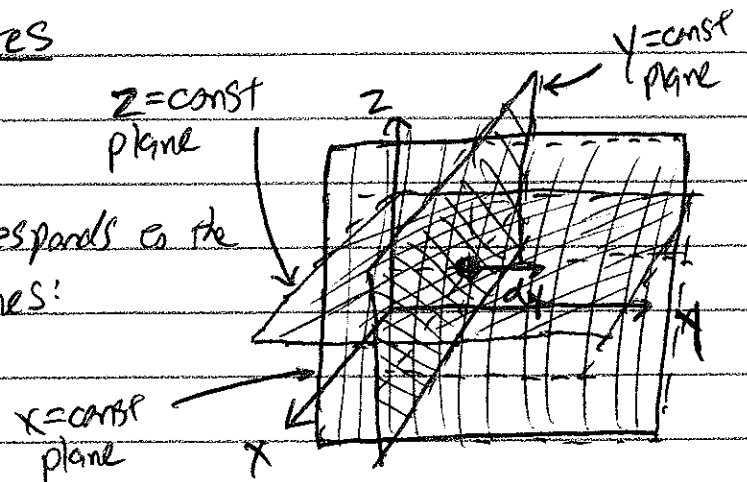
1. Often equations can be simplified if the symmetry of a problem is used so select an optimal coordinate system.
  - a. Ex: Central force problems (gravity)  $\Rightarrow$  Spherical Polar coords.
2. Adapting non-Cartesian coordinates adds complexity to vector operators.

### B. Orthogonal Coordinates

#### 1. Cartesian Coordinates:

- a. A point  $(x_0, y_0, z_0)$  corresponds to the intersection of three planes:

$$x = x_0, \quad y = y_0, \quad z = z_0$$



- b. A change in  $y$  corresponds to a displacement normal to plane of constant  $y = y_0$ .
- c. Planes are mutually perpendicular (orthogonal) at each point.

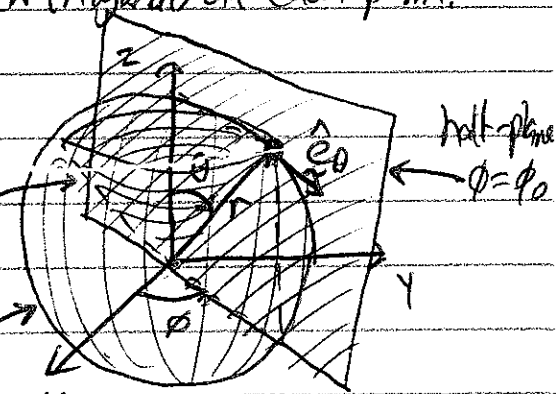
#### 2. Spherical Polar Coordinates

$$(r_0, \theta_0, \phi_0)$$

- a. General coordinates need not be lengths
- b. Direction normal to

a surface depends on position

half-cone  
 $\theta = \theta_0$   
 sphere of radius  $r = r_0$



I.B. Continued)

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3. Unit vectors at a particular position,  $(r, \theta, \phi)$ , given by  $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$  depend on position.

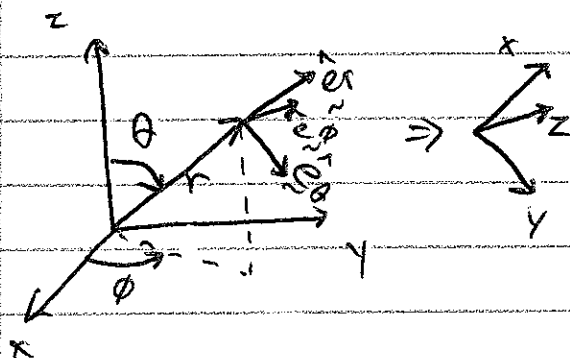
b. But, at any point, the unit vectors are mutually perpendicular (orthogonal)

4. For an arbitrary vector  $\underline{V}$ ,

$$\underline{V}(r) = V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_\phi \hat{e}_\phi$$

a. Also, note  $V_r, V_\theta, V_\phi$  have

the same units  $\Rightarrow V_\theta$  is not  $\theta!$  directions depend on  $r$ .



b. Can think of it as a local Cartesian system  $\Rightarrow V_r, V_\theta, V_\phi$  are components in that system.

5. At same point  $r$ ,  $\underline{V}(r) \cdot \underline{W}(r) = V_r W_r + V_\theta W_\theta + V_\phi W_\phi$

a. Can only do vector algebra on two vectors at same point  $r$ .

6. NOTE: a. Position  $\underline{r} \neq r \hat{e}_r + \theta \hat{e}_\theta + \phi \hat{e}_\phi$  wrong dimensions!  
b. Instead  $\underline{r} = r \hat{e}_r$ , where  $\hat{e}_r$  depends on  $(\theta, \phi)$ .

### C. Arbitrary Curvilinear System

1. For coordinates  $(q_1, q_2, q_3)$ , we have  $x(q_1, q_2, q_3)$ , so

$$dx = \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3$$

2. For  $(dr)^2 = (dx)^2 + (dy)^2 + (dz)^2$ , we obtain

Metric, or Riemannian

$$(dr)^2 = \sum_{ij} g_{ij} dq_i dq_j$$

where

$$g_{ij} = \frac{\partial x}{\partial q_i} \frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i} \frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i} \frac{\partial z}{\partial q_j}$$

I. C. (Continued)

HWK 3

3. a. We can interpret  $g_{ij}$  as the product of vector in  $q_i$  direction with components  $(\frac{\partial x}{\partial q_i}, \frac{\partial y}{\partial q_i}, \frac{\partial z}{\partial q_i})$  and vector in  $q_j$  direction with  $(\frac{\partial x}{\partial q_j}, \frac{\partial y}{\partial q_j}, \frac{\partial z}{\partial q_j})$

b. If  $q_i$  coordinates are orthogonal, then  $g_{ij} = 0$  if  $i \neq j$ .  $\Rightarrow$  Orthogonal Coordinate System

#### 4. General Results for Orthogonal Coordinate Systems

a.  $(dr)^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2$

where  $h_i^2 = \left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2$

b. For a Cartesian coordinate  $x_i$ ,  $dx_i = h_i dq_i$  dimension of length

$\Rightarrow \frac{dx}{dq_i} = h_i \hat{e}_i$  or  $\boxed{dr = h_1 dq_1 \hat{e}_1 + h_2 dq_2 \hat{e}_2 + h_3 dq_3 \hat{e}_3}$

#### c. Integration:

i. Line Integral  $\int_C \vec{V} \cdot d\vec{r} = \int_C V_i h_i dq_i$

ii. Surface Integral  $\int_S \vec{V} \cdot d\vec{r} = \int_S V_1 h_2 h_3 dq_2 dq_3 + \int_S V_2 h_3 h_1 dq_3 dq_1 + \int_S V_3 h_1 h_2 dq_1 dq_2$

iii. Volume Integral  $\int_V dV(q_1, q_2, q_3) = \underbrace{h_1 h_2 h_3}_{=dr} dq_1 dq_2 dq_3$

#### d. Differentiation:

i. Gradient:  $\nabla \phi(q_1, q_2, q_3) = \hat{e}_1 \frac{1}{h_1} \frac{\partial \phi}{\partial q_1} + \hat{e}_2 \frac{1}{h_2} \frac{\partial \phi}{\partial q_2} + \hat{e}_3 \frac{1}{h_3} \frac{\partial \phi}{\partial q_3}$

Thus  $\nabla = \hat{e}_1 \frac{1}{h_1} \frac{\partial}{\partial q_1} + \hat{e}_2 \frac{1}{h_2} \frac{\partial}{\partial q_2} + \hat{e}_3 \frac{1}{h_3} \frac{\partial}{\partial q_3}$

ii. Divergence:  $\nabla \cdot \vec{V}(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial q_1} (V_1 h_2 h_3) + \frac{\partial}{\partial q_2} (V_2 h_3 h_1) + \frac{\partial}{\partial q_3} (V_3 h_1 h_2) \right]$

## I. C4d (Continued)

iii. Laplacian  $\nabla^2 \phi = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial z_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial z_1} \right) + \frac{\partial}{\partial z_2} \left( \frac{h_3 h_1}{h_2} \frac{\partial \phi}{\partial z_2} \right) + \frac{\partial}{\partial z_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial z_3} \right) \right]$  Hanes (4)

NOTE: No cross derivatives because coordinates are orthogonal!

### iv. Curl:

$$\nabla \times \underline{B} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{e}_1 h_1 & \hat{e}_2 h_2 & \hat{e}_3 h_3 \\ \frac{\partial}{\partial z_1} & \frac{\partial}{\partial z_2} & \frac{\partial}{\partial z_3} \\ h_1 B_1 & h_2 B_2 & h_3 B_3 \end{vmatrix}$$

## II. Circular Cylindrical Coordinates $(\rho, \phi, z)$

### A. Formulas

1. a.  $\rho = (x^2 + y^2)^{\frac{1}{2}}$

b.  $x = \rho \cos \phi$

$\phi = \tan^{-1} \left( \frac{y}{x} \right)$

$y = \rho \sin \phi$

$z$

$z$

Range:

$$0 \leq \rho < \infty$$

$$0 \leq \phi < 2\pi$$

$$-\infty < z < \infty$$

### 2a. Position Coordinates:

$$\underline{r} = \rho \hat{e}_\rho + z \hat{e}_z$$

### b. Vector:

$$\underline{V} = V_\rho \hat{e}_\rho + V_\phi \hat{e}_\phi + V_z \hat{e}_z$$

### 3. Scale Factors:

$$h_\rho = 1 \quad h_\phi = \rho \quad h_z = 1$$

### 4. Integration:

a.  $d\underline{r} = \hat{e}_\rho d\rho + \rho \hat{e}_\phi d\phi + \hat{e}_z dz$

b.  $d\underline{\Omega} = \rho \hat{e}_\rho d\phi dz + \hat{e}_\phi \rho d\rho dz + \rho \hat{e}_z d\rho d\phi$

c.  $d\underline{\tau} = \rho d\rho d\phi dz$

### 5. Vector differential operators:

a.  $\nabla \Psi(\rho, \phi, z) = \hat{e}_\rho \frac{\partial \Psi}{\partial \rho} + \hat{e}_\phi \frac{1}{\rho} \frac{\partial \Psi}{\partial \phi} + \hat{e}_z \frac{\partial \Psi}{\partial z}$

b.  $\nabla \cdot \underline{V} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho V_\rho) + \frac{1}{\rho} \frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}$

II A (Continued)

Hawes (5)

$$c. \nabla^2 \psi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

$$d. \nabla_{\hat{r}} \cdot \underline{\hat{r}} = \frac{1}{\rho} \begin{vmatrix} \hat{e}_\rho & \rho \hat{e}_\phi & \hat{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ V_\rho & \rho V_\phi & V_z \end{vmatrix}$$

$$e. \nabla^2 \underline{\hat{r}} = \left[ \nabla^2 V_\rho - \frac{1}{\rho^2} V_\rho - \frac{2}{\rho^2} \frac{\partial V_\phi}{\partial \phi} \right] \hat{e}_\rho + \left[ \nabla^2 V_\phi - \frac{1}{\rho^2} V_\phi + \frac{2}{\rho^2} \frac{\partial V_\rho}{\partial \rho} \right] \hat{e}_\phi + \nabla^2 V_z \hat{e}_z$$

B. Examples:

1. Ex: Kepler's Area Law: (Kepler's 2nd Law)

a. The radius vector of a planet sweeps out equal areas in equal times.

b.  $\underline{F}_g = f(r) \hat{e}_r$  (taking  $z=0$ ,  $r=\rho$ )

c. Torque about origin must vanish  $\underline{\tau} = \underline{r} \times \underline{F} = r \hat{e}_r \times f(r) \hat{e}_r = 0$

d. Thus,  $\underline{L} = \underline{r} \times \underline{v} = \underline{r} \times \frac{d\underline{r}}{dt} = \text{constant}$ .

$$e. d\underline{r} = \hat{e}_\rho d\rho + \rho \hat{e}_\phi d\phi + \hat{e}_z dz$$

$$\Rightarrow \frac{d\underline{r}}{dt} = \frac{d\rho}{dt} \hat{e}_\rho + \rho \frac{d\phi}{dt} \hat{e}_\phi = \dot{\rho} \hat{e}_\rho + \rho \dot{\phi} \hat{e}_\phi \quad \hat{e}_\rho \times \hat{e}_\phi = \hat{e}_z$$

$$f. \text{Thus } \underline{L} = \underline{r} \times \frac{d\underline{r}}{dt} = \rho \hat{e}_\rho \times [\dot{\rho} \hat{e}_\rho + \rho \dot{\phi} \hat{e}_\phi] = \rho^2 \dot{\phi} \hat{e}_z$$

g. So  $\rho^2 \dot{\phi} = \text{constant!}$

b. But  $\rho^2 \dot{\phi} = 2 \frac{dA}{dt}$  (A is area swept out)  $\Rightarrow$  Area swept out constant!

2. Ex: Compute Nonlinear term in Navier-Stokes Eqs for hydrodynamics

$$\nabla \times [\underline{v} \times (\nabla \times \underline{v})] \quad \text{for } \underline{v} = \hat{e}_z v(\rho) \quad (\hat{z} \text{ direction})$$

$$a. \nabla_{\hat{r}} \cdot \underline{\hat{r}} = \frac{1}{\rho} \begin{vmatrix} \hat{e}_\rho & \rho \hat{e}_\phi & \hat{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & v(\rho) \end{vmatrix} = \frac{1}{\rho} \left[ \hat{e}_\rho \frac{\partial v(\rho)}{\partial \rho} - \hat{e}_\phi \rho \frac{\partial v(\rho)}{\partial \rho} \right] = -\hat{e}_\phi \frac{\partial v(\rho)}{\partial \rho}$$

## II. B. 2 (Continued)

$$b. \nabla \times (\nabla \times \underline{v}) = \begin{vmatrix} \hat{e}_\rho & \hat{e}_\phi & \hat{e}_z \\ 0 & 0 & \tilde{v} \\ 0 & -\frac{\partial v}{\partial \rho} & 0 \end{vmatrix} = -\hat{e}_\phi (v) \left( \frac{\partial v}{\partial \rho} \right) = \hat{e}_\phi v \frac{dv}{d\rho} \quad \text{Haves 6}$$

$$c. \nabla \times [\nabla \times (\nabla \times \underline{v})] = \frac{1}{\rho} \begin{vmatrix} \hat{e}_\rho & \rho \hat{e}_\phi & \hat{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \frac{v \partial v}{\partial \rho} & 0 & 0 \end{vmatrix} = \frac{1}{\rho} \left[ \rho \hat{e}_\phi \frac{\partial}{\partial z} \left( \frac{\partial v}{\partial \rho} \right) - \hat{e}_z \frac{\partial}{\partial \phi} \left( \frac{v \partial v}{\partial \rho} \right) \right] = 0!$$

## III. Spherical Polar Coordinates $(r, \theta, \phi)$

A. Formulas:

$$1. a. r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$$

$$b. x = r \sin \theta \cos \phi$$

$$\theta = \cos^{-1} \left( \frac{z}{r} \right)$$

$$y = r \sin \theta \sin \phi$$

$$\phi = \tan^{-1} \left( \frac{y}{x} \right)$$

$$z = r \cos \theta$$

Range:

$$0 \leq r < \infty$$

$$0 \leq \theta \leq \pi$$

$$0 \leq \phi < 2\pi$$

2a. Position Coordinate:  $\underline{r} = r \hat{e}_r$  [where  $\hat{e}_r = f(\theta, \phi)$ ]

b. Vector  $\underline{v} = v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_\phi \hat{e}_\phi$

3. Scale Factors:

$$h_r = 1 \quad h_\theta = r \quad h_\phi = r \sin \theta$$

4. Integration a.  $d\underline{r} = \hat{e}_r dr + r \hat{e}_\theta d\theta + r \sin \theta \hat{e}_\phi d\phi$

$$b. d\underline{v} = r^2 \sin \theta \hat{e}_r d\theta d\phi + r \sin \theta \hat{e}_\theta dr d\phi + r \hat{e}_\phi dr d\theta$$

$$c. d\tau = r^2 \sin \theta dr d\theta d\phi \quad \text{where } d\Omega = d\theta d\phi$$

$$\text{and } \int d\Omega = 4\pi$$

5. Note: All unit vectors depend on position, so

$$\frac{\partial \hat{e}_i}{\partial x_j} \neq 0! \quad (\text{unlike Cartesian})$$

### III A. (Continued)

G. Vector Differential Operators: a.  $\nabla \Phi(r, \theta, \phi) = \hat{e}_r \frac{\partial \Phi}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial \Phi}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin \theta} \frac{\partial \Phi}{\partial \phi}$  Hanes (7)

$$b. \nabla \cdot \vec{V} = \frac{1}{r^2 \sin \theta} \left[ \sin \theta \frac{\partial}{\partial r} (r^2 V_r) + r \frac{\partial}{\partial \theta} (\sin \theta V_\theta) + r \frac{\partial V_\phi}{\partial \phi} \right]$$

$$c. \nabla^2 \Phi = \frac{1}{r^2 \sin \theta} \left[ \sin \theta \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 \Phi}{\partial \phi^2} \right]$$

$$d. \nabla \times \vec{V} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{e}_r & r \hat{e}_\theta & r \sin \theta \hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ V_r & r V_\theta & r \sin \theta V_\phi \end{vmatrix}$$

$$e. \nabla^2 \vec{V} = \left[ \nabla^2 V_r - \frac{2}{r^2} V_r - \frac{2}{r^2} \cot \theta V_\theta - \frac{2}{r^2} \frac{\partial V_\theta}{\partial \theta} - \frac{2}{r^2 \sin \theta} \frac{\partial V_\phi}{\partial \phi} \right] \hat{e}_r$$

$$+ \left[ \nabla^2 V_\theta - \frac{1}{r^2 \sin^2 \theta} V_\theta + \frac{2}{r^2} \frac{\partial V_r}{\partial \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial V_\phi}{\partial \phi} \right] \hat{e}_\theta$$

$$+ \left[ \nabla^2 V_\phi - \frac{1}{r^2 \sin^2 \theta} V_\phi + \frac{2}{r^2 \sin \theta} \frac{\partial V_r}{\partial \phi} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial V_\theta}{\partial \phi} \right] \hat{e}_\phi$$

### B. Examples:

1. Ex: Central Force: For a potential  $\Phi(r)$ , compute  $\nabla \Phi$ ,  $\nabla^2 \Phi$ .

$$a. \nabla \Phi(r) = \hat{e}_r \frac{\partial \Phi}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial \Phi}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin \theta} \frac{\partial \Phi}{\partial \phi} = \boxed{\hat{e}_r \frac{\partial \Phi}{\partial r}}$$

$$b. \nabla^2 \Phi(r) = \nabla \cdot [\nabla \Phi(r)] = \frac{1}{r^2 \sin \theta} \left[ \sin \theta \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) \right] = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right)$$

$$= \frac{1}{r^2} \left[ 2r \frac{\partial \Phi}{\partial r} + r^2 \frac{\partial^2 \Phi}{\partial r^2} \right] = \boxed{\frac{2}{r} \frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial r^2}}$$

2. Ex: Magnetic Vector Potential: A loop in the  $xy$  plane with a steady current yields a vector potential  $\vec{A}(r, \theta, \phi) = A_\phi(r, \theta) \hat{e}_\phi$ .  
Compute the current in terms of  $A_\phi$ .

### III. B. 2. (Continued)

a. For steady state:  $\mu_0 \underline{J} = \nabla \times \underline{B} - \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} \Rightarrow \underline{J} = \frac{1}{\mu_0} \nabla \times \underline{B}$  Hawes (8)

b.  $\underline{B} = \nabla \times \underline{A}$ , so  $\underline{J} = \frac{1}{\mu_0} [\nabla \times (\nabla \times \underline{A})]$  where  $\underline{A} = A_\phi(r, \theta) \hat{e}_\phi$

c. 
$$\nabla \times \underline{A} = \frac{1}{r \sin \theta} \begin{vmatrix} \hat{e}_r & r \hat{e}_\theta & r \sin \theta \hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r \sin \theta A_\phi(r, \theta) \end{vmatrix} = \frac{1}{r \sin \theta} \left[ \hat{e}_r \frac{\partial}{\partial \theta} [r \sin \theta A_\phi] - r \hat{e}_\theta \frac{\partial}{\partial r} [r \sin \theta A_\phi] \right]$$

$$= \frac{\hat{e}_r}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\phi) - \frac{\hat{e}_\theta}{r} \frac{\partial}{\partial r} (r A_\phi)$$

d. 
$$\nabla \times (\nabla \times \underline{A}) = \frac{1}{r \sin \theta} \begin{vmatrix} \hat{e}_r & r \hat{e}_\theta & r \sin \theta \hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\phi) & -\frac{\partial}{\partial r} (r A_\phi) & 0 \end{vmatrix} = \frac{1}{r \sin \theta} \left\{ r \sin \theta \hat{e}_r \left[ \frac{\partial^2}{\partial r^2} (r A_\phi) \right] \right.$$

$$\left. - r \sin \theta \hat{e}_\theta \frac{\partial}{\partial \theta} \left[ \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\phi) \right] \right\} = \hat{e}_\phi \left\{ -\frac{1}{r} \frac{\partial^2}{\partial r^2} (r A_\phi) - \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\phi) \right] \right\}$$

## IV. Reflection in Spherical Coordinates

### A. Reflection

1.

$r$	$\rightarrow$	$r$
$\theta$	$\rightarrow$	$\pi - \theta$
$\phi$	$\rightarrow$	$\pi + \phi$

### 2. Vector Components

$$\begin{pmatrix} A'_r \\ A'_\theta \\ A'_\phi \end{pmatrix} = \begin{pmatrix} A_r \\ -A_\theta \\ A_\phi \end{pmatrix} \quad (\text{simply changes } A_\theta \text{ sign})$$