PHYS:4762 Mathematical Methods of Physics II

Lecture #1: Complex Variable Theory and Cauchy's Integral Theorem

I. Complex Variables and Functions

A. Basics

1. Complex Variable Theory is a powerful & widely used analysis tool.
2. In physics, complex variables play a vital role in theory for electromagnetism, waves, quantum mechanics, solution of differential equations, and evaluating key integrals.

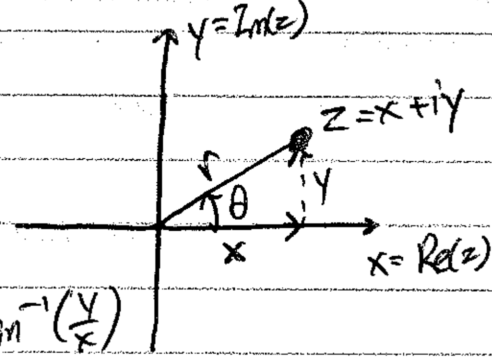
B. Important Properties

1. Lecture #2 from PHYS:4761 reviews some basic properties. Here we highlight some of the most important properties.

2\. a. Complex Variable: $z = x + iy$ (x & y are real), $i^2 = -1$

b. Complex Conjugate: $z^* = x - iy$ (change sign of i)

c. $|z|^2 = z z^*$ is real



3\. Cartesian and polar representations

a. $z = x + iy = r\cos\theta + i r\sin\theta = re^{i\theta}$

b. $x = r\cos\theta$, $y = r\sin\theta$; $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

4\. In general, $\boxed{e^{iz} = \cos z + i\sin z}$

5\. Complex Functions, $f(z)$

a. Real and Imaginary parts: $f(z) = U(x,y) + iV(x,y)$

b. Multivalued Functions: Since $e^{i2\pi n} = 1$ for any integer n, roots of complex variables are multivalued.

i) $z^{\frac{1}{m}}$ has m complex values.

I. B.5.b. (Continued)

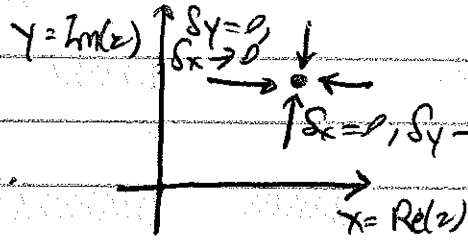
ii) <u>Logarithm</u>: $\ln(z) = \ln(re^{i\theta}) = \ln r + i(\theta + 2\pi n)$ ← multivalued.

II. <u>Cauchy-Riemann Conditions</u>

A. <u>Differentiating Complex Functions</u>

1. Take $f(z) = u(x,y) + iv(x,y)$ for $z = x + iy$
2. <u>Def: Derivative</u> $\displaystyle \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{(z+\delta z) - z} = \lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z} = \frac{df}{dz} = f'(z)$

a. Derivative exists <u>only</u> if limit is independent of approach to z.



⇒ Leads to <u>significant restrictions</u> on $u(x,y)$ & $v(x,y)$!

3. Take $\delta z = \delta x + i\delta y$ and $\delta f = \delta u + i\delta v$.

Thus $\displaystyle \frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}$

a. Consider approach $\delta x \to 0$ with $\delta y = 0$.

$$\Rightarrow \lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta z \to 0}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

b. Alternative approach $\delta y \to 0$ with $\delta x = 0$

$$\Rightarrow \lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta z \to 0}\left(-i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y}\right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

c. Since these must be equal, it requires

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Cauchy-Riemann Conditions

II. A. (Continued)

4. $\boxed{\text{For } \frac{df}{dz} \text{ to exist, Cauchy-Riemann conditions must hold}}$

5. Conversely, if Cauchy-Riemann conditions hold and partial derivatives of $U(x,y)$ and $V(x,y)$ are continuous, $\frac{df}{dz}$ exists.

B. Analytic Functions

1. Def: Analytic: A function $f(z)$ is analytic if it is differentiable and single-valued over a region of the complex plane.

2. Def: Entire: A function $f(z)$ that is analytic over the entire (finite) complex plane.

3. Def: Singular Point: If $f'(z)$ does not exist at $z = z_0$, then z_0 is a singular point.

4. Ex: Is $f(z) = z^2$ analytic?

a. $f(z) = z^2 = (x+iy)(x+iy) = x^2 - y^2 + i\,2xy$

b. Thus, $U(x,y) = x^2 - y^2$ and $V(x,y) = 2xy$

c. Test Cauchy-Riemann Conditions:

$$\frac{\partial U}{\partial x} = 2x = \frac{\partial V}{\partial y} \checkmark \qquad \frac{\partial U}{\partial y} = -2y = -\frac{\partial V}{\partial x} \checkmark$$

$\longrightarrow f(z) = z^2$ is differentiable

d. The partial derivatives are continuous also

e. Thus, $f(z) = z^2$ is analytic (also an entire function).

5. Ex: $f(z) = z^*$

a. $f(z) = z^* = x - iy \Rightarrow U = x,\ V = -y$

II. B.5. (Continued)

b. Cauchy-Riemann: $\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1 \rightarrow$ Not differentiable $\rightarrow$ <u>Not analytic</u>

<u>C. More Characteristics of Analytic Functions</u>

1. The derivative of a real function is a <u>local</u> characteristic, but for analytic complex functions, the existence of a derivative implies much more.

2. <u>Satisfaction of Laplace's Equation</u>

a. Both $u(x,y)$ and $v(x,y)$ must satisfy $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$.

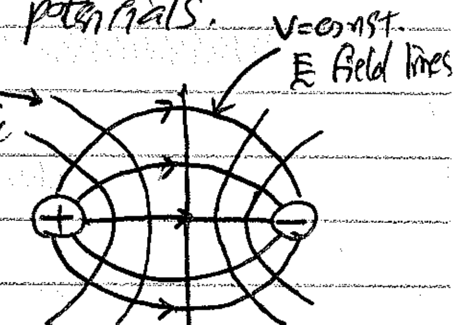
b. From Cauchy-Riemann Conditions, $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$,

So $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Same for $v(x,y)$.

c. Closely related to solutions of electrostatic potentials.

3. <u>Orthogonal Characteristic Curves</u>

a. Curves of $u(x,y) = $ const are everywhere orthogonal to curves of $v(x,y) = $ const.



4. Analytic functions have not only first derivatives, but <u>derivatives of all higher orders</u>.

<u>D. Derivatives of Analytic Functions</u>

1. $\boxed{\text{Complex Differentiation follows the same rules as those for real variables.}}$

a. <u>Product:</u> $[f(z)g(z)]' = f'(z)g(z) + f(z)g'(z)$

b. $$\frac{dz^n}{dz} = n z^{n-1}$$

II. (Continued)

E. Singularity at Infinity

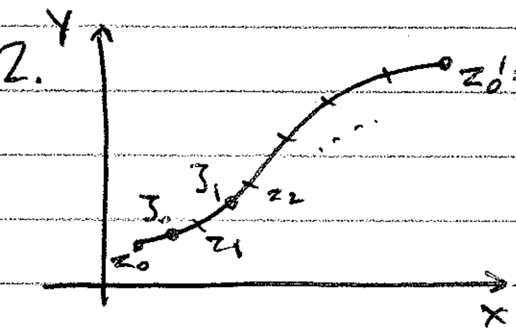
1. In complex variable theory, infinity is regarded as a single point.
   a. Make variable change from z to $w = \frac{1}{z}$.
   b. Thus, for R large, $z = -R$ lies close to $z = +R$.

2. Entire functions, such as z or z^2, have singular point at $z=\infty$.
   a. $z = \frac{1}{w}$ as $w \to 0 \Rightarrow z$ is singular at $z = \infty$!

III. Cauchy's Integral Theorem

A. Contour Integrals

1. Integral of a complex variable requires specification of the path (contour) in the complex plane.

2. 

   a. $S_n = \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1})$

   b. $\lim_{n\to\infty} S_n = \int_{z_0}^{z_0'} f(z)\,dz = \int_C f(z)\,dz$ Contour integral.

3. In terms of real integrals,

$$\int_{z_1}^{z_2} f(z)\,dz = \int_{x_1,y_1}^{x_2,y_2} [u(x,y) + iv(x,y)][dx + i\,dy]$$

$$= \int_{x_1,y_1}^{x_2,y_2} [u(x,y)dx - v(x,y)dy] + i\int [v(x,y)dx + u(x,y)dy]$$

Complex integral as complex sum of real integrals.

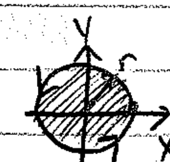
4. Closed Contour: $\oint_C f(z)\,dz$ traversed CCW (RH rule).

B. Cauchy's Integral Theorem

If $f(z)$ is an analytic function within a simply connected region and if C is a closed contour in that region, then

$$\oint_C f(z)\,dz = 0$$

1. Simply connected if every closed curve can be shrunk to a point within the region (region with no holes).
2. Contour must be within analytic region (not on boundary).
3. Ex: Evaluate $\oint_C z^n dz$



a. Take C a circle of radius r CCW around origin.

b. Take $z = re^{i\theta}$ and $dz = ire^{i\theta}d\theta$ (since r=const).

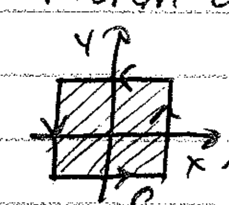
c. Thus, $\displaystyle\oint_C z^n dz = ir^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}d\theta = ir^{n+1}\left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi} = 0.$ for $n \neq -1$.

d. But, if $n=-1$,

$$\oint_C z^n dz = \oint_C \frac{dz}{z} = i\int_0^{2\pi}\frac{\cancel{re^{i\theta}}\,d\theta}{\cancel{re^{i\theta}}} = i\int_0^{2\pi} d\theta = 2\pi i \neq 0.$$

e. Cauchy's integral theorem requires $f(z)$ to be analytic throughout region, but for $n<0$, z^n is singular at $z=0$!

f. For all $n \geq 0$, Cauchy's theorem applies → as we have seen!

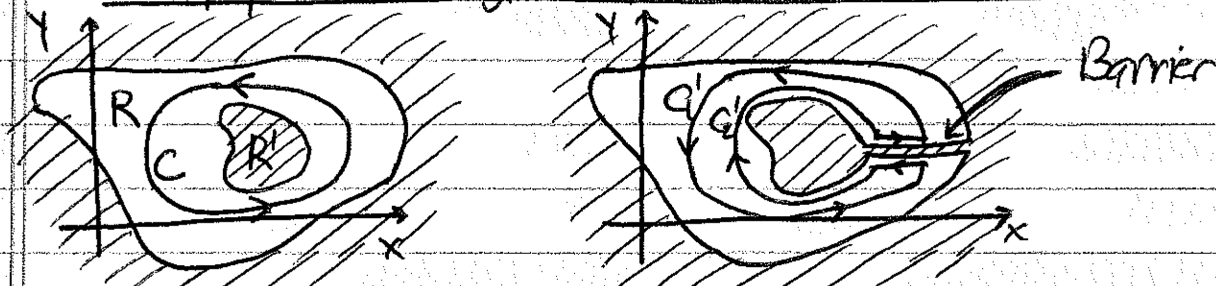
4. For a different path  , the result is the same (See text for worked example).

III. B. (Continued)

5. Proof uses Stoke's Theorem to convert the real and imaginary parts of contour integral into a form that yields zero if Cauchy-Riemann conditions are satisfied,

$$\oint_C f(z)dz = -\int_A \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dxdy + i\int_A \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy = 0.$$

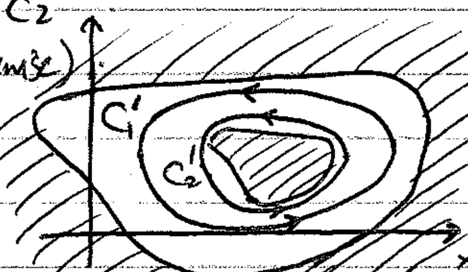
C. Multiply Connected Regions and the Deformation of Paths



1. $$\oint_C f(z)dz = \oint_{C_1'} f(z)dz - \underset{\substack{\uparrow \\ \text{(Clockwise)}}}{\oint_{C_2'}} f(z)dz = 0$$ ← By Cauchy Integral Theorem over simply connected region.

2.

$$\boxed{\oint_{C_1'} f(z)dz = \oint_{C_2'} f(z)dz}$$
(CCW)



3. Principle of Deformation of Paths

> The integral of an analytic function over a closed path is unchanged for any possible continuous deformations within analytic region

4. From example of $\oint_C z^n dz$ and deformation of paths,

> The integral of $(z-z_0)^n$ around any counterclockwise path encircling z_0,
> $$\oint_C (z-z_0)^n dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$$