Lecture #12 Legendre Functions: Orthogonality, Generating Function, and Associated Legendre Equation

I. Orthogonality

A. Legendre Polynomials

1. Legendre Equation: $(1-x^2)P''(x)-2xP'(x)+\ell(\ell+1)P(x)=0$
   a. Self-adjoint already, so $w(x)=1$

2. Orthogonality Condition: $\boxed{\int_{-1}^{1} P_n(x)P_m(x)=0}$, $n\neq m$

   or using $x=\cos\theta$ $\boxed{\int_0^{\pi} P_n(\cos\theta)P_m(\cos\theta)\sin\theta\,d\theta=0}$ $n\neq m$

3. Normalization:
   a. Generating Function $[g(x,t)]^2=(1-2xt+t^2)^{-1}=\left[\sum_{n=0}^{\infty} P_n(x)t^n\right]^2$

   b. Integrate from $\int_{-1}^{1}dx$ (Cross terms vanish due to orthogonality)

$$\int_{-1}^{1}\frac{dx}{1-2xt+t^2}=\sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1}[P_n(x)]^2dx$$

   c. By substituting $y=(1-2xt+t^2)$, one may integrate LHS. Then, expanding result as a power series in t, we may equate coefficients to obtain the normalization.

$$\Rightarrow \boxed{\int_{-1}^{1}P_n(x)P_m(x)dx=\frac{2\delta_{nm}}{2n+1}}$$ ← Not orthonormal.

B. Expansion in Legendre Series

1. $\boxed{f(x)=\sum_{n=0}^{\infty} a_n P_n(x)}$, $-1\leq x\leq 1$

where $\boxed{a_n=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)dx}$

   a. Expansion is unique

2. General Solution of Laplace's Eq $\nabla^2\psi = 0$ in spherical coordinates

$$\psi(r,\theta,\phi) = \sum_{l,m} \left(A_{lm} r^{l} + B_{lm} r^{-l-1}\right) P_l^m(\cos\theta)\left(A'_{lm}\sin m\phi + B'_{lm}\cos m\phi\right)$$

a. l must be an integer (to avoid divergence at poles $\theta=0$ & $\theta=\pi$)

3. Axisymmetric Solutions ($m=0$) (no ϕ dependence)

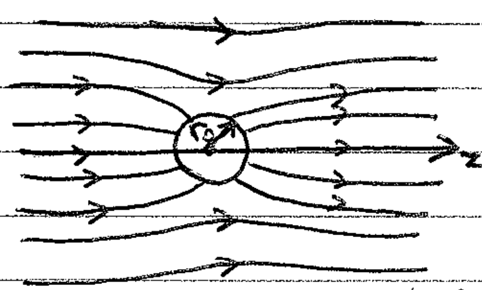
$$\boxed{\psi(r,\theta) = \sum_{l=0}^{\infty} \left(a_l r^{l} + b_l r^{-l-1}\right) P_l(\cos\theta)}$$

4. Spherical Boundary (at $r=r_0$): a) Interior Solution: $\psi(r,\theta) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta)$ $r \le r_0$

b) Exterior Solution: $\psi(r,\theta) = \sum_{l=0}^{\infty} b_l r^{-l-1} P_l(\cos\theta)$ $r \ge r_0$

C. Ex: Conducting Sphere in Uniform Electric Field

1.



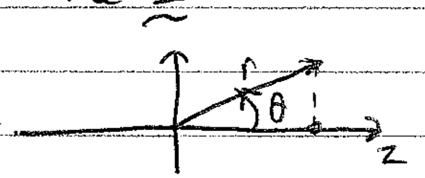
a. Background field $\underset{\sim}{E} = E_0 \hat{z}$

b. Align $\hat{z}$ axis to $\underset{\sim}{E}$, $\rightarrow$ axisymmetric $\Rightarrow$ no ϕ dependence.

2. For $r > r_0$, solution of Laplace's Eq $\nabla^2\psi = 0$

3. Since $\underset{\sim}{E} = -\nabla\psi = -\frac{\partial\psi}{\partial z}\hat{z} = E_0\hat{z} \Rightarrow \boxed{\psi = -E_0 z + \psi_0}$ Potential of Background field.

4.



$z = r\cos\theta$, so $\psi = -E_0 r\cos\theta + \psi_0$

5. General Solution: $\psi(r,\theta) = \sum_{l=0}^{\infty} \left(a_l r^l + b_l r^{-l-1}\right) P_l(\cos\theta)$

6. <u>Boundary condition at $r \to \infty$:</u>

a. Effect of sphere must be local, so $\Psi(r\to\infty) = -E_0 r\cos\theta + \Psi_0$

b. Since $P_1(\cos\theta) = \cos\theta$, this means $a_1 = -E_0$, $a_0 = \Psi_0$
$a_n = 0$, $n > 2$

c. No constraint on b_ℓ from $r \to \infty$.

$$\Rightarrow \Psi(r,\theta) = \Psi_0 - E_0 r P_1(\cos\theta) + \sum_{\ell=0}^{\infty} b_\ell r^{-\ell-1} P_\ell(\cos\theta)$$

7. <u>Boundary Condition at $r = r_0$:</u>

a. For a conducting sphere $\Psi(r=r_0) = \text{const} = \Psi_0$.

b. Since potential may always have additive constant, we may set $\Psi_0 = 0$.

c. Thus

$$\Psi(r_0,\theta) = -E_0 r_0 P_1(\cos\theta) + \frac{b_1}{r_0^2} P_1(\cos\theta) + \frac{b_0}{r_0} + \sum_{\ell=2}^{\infty} b_\ell r^{-\ell-1} P_\ell(\cos\theta)$$
$$= 0$$

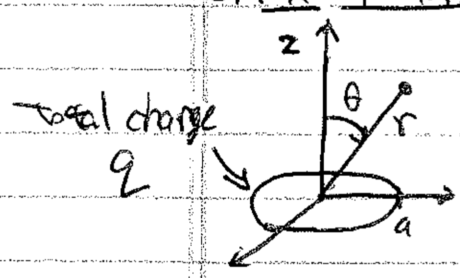
d. So $$0 = \underset{\substack{\uparrow \\ 1 = P_0(\cos\theta)}}{\frac{b_0}{r_0}} + \left(\frac{b_1}{r_0^2} - E_0 r_0\right) P_1(\cos\theta) + \sum_{\ell=2}^{\infty} b_\ell r^{-\ell-1} P_\ell(\cos\theta)$$

e. Setting all coefficients of $P_n(\cos\theta)$ equal, we obtain
$b_0 = 0$
$b_1 = E_0 r_0^3$
$b_n = 0 \quad n \geq 2$

8. Final Solution:
$$\Psi(r,\theta) = P_1(\cos\theta)\left[-E_0 r + \frac{E_0 r_0^3}{r^2}\right] = -E_0 \underbrace{r\cos\theta}_{=z}\left(1 - \frac{r_0^3}{r^3}\right)$$

$$\boxed{\Psi(r,\theta) = -E_0 z\left(1 - \frac{r_0^3}{r^3}\right)}$$

D. <u>Ex: Potential due to a Charged Ring</u>



1. For $r > a$, solve $\nabla^2\Psi = 0$

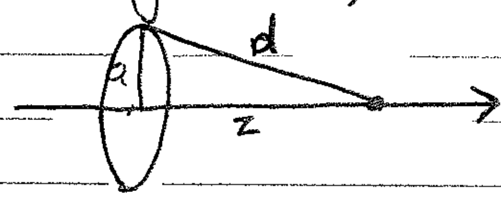
2. General Solution: $$\Psi(r,\theta) = \sum_{\ell=0}^{\infty} b_\ell r^{-\ell-1} P_\ell(\cos\theta)$$

since $\Psi \to 0$ as $r \to \infty$ (thus $a_\ell = 0$)

I. D. (Continued) Hawes (4)

3. We may use solution along z-axis to determine b_ℓ.

a. Along z-axis, $r=z$, $\theta=0$ (so $P_\ell(\cos\theta)=1$ for all ℓ).



b. All charge at distance $d=\sqrt{a^2+z^2}$

$$\Rightarrow \Psi(z,0)=\frac{q}{4\pi\varepsilon_0}\frac{1}{(a^2+z^2)^{\frac{1}{2}}}$$

c. We can expand denominator in $\left(\frac{a}{z}\right)^2$ and equate powers of z^{-n},

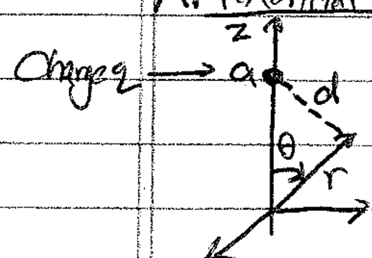
$$\Psi(z,0)=\frac{q}{4\pi\varepsilon_0 z}\sum_{s=0}^{\infty}(-1)^s\frac{(2s-1)!!}{(2s)!!}\left(\frac{a}{z}\right)^{2s}=\frac{1}{z}\sum_{\ell=0}^{\infty}\frac{b_\ell}{z^\ell}$$

$$\Rightarrow b_{2\ell}=\frac{q\,a^{2\ell}}{4\pi\varepsilon_0}(-1)^\ell\frac{(2\ell-1)!!}{(2\ell)!!},\quad b_{2\ell+1}=0$$

4. Thus $\boxed{\Psi(r,\theta)=\frac{q}{4\pi\varepsilon_0 r}\sum_{\ell=0}^{\infty}(-1)^\ell\frac{(2\ell-1)!!}{(2\ell)!!}\left(\frac{a}{r}\right)^{2\ell}P_\ell(\cos\theta)}$, $r>a$

II. Physical Interpretation of Generating Function

A. Potential of a Charge on z-axis



1. $\Psi(r,\theta)=\frac{q}{4\pi\varepsilon_0}\frac{1}{d}=\frac{q}{4\pi\varepsilon_0}\frac{1}{(a^2-2ar\cos\theta+r^2)^{\frac{1}{2}}}$

a. Law of Cosines:
$d^2=r^2+a^2-2ar\cos\theta$

2. Generating Function: $g(x,t)=\frac{1}{(1-2xt+t^2)^{\frac{1}{2}}}$ $\quad x=\cos\theta$, $t=\frac{a}{r}$

a. Take $\Psi(r,\theta)=\frac{q}{4\pi\varepsilon_0 r}\frac{1}{\left(1-2\frac{a}{r}\cos\theta+\frac{a^2}{r^2}\right)^{\frac{1}{2}}}=\frac{q}{4\pi\varepsilon_0 r}\,g\!\left(\cos\theta,\frac{a}{r}\right)$

b. Thus $\Psi(r,\theta)=\frac{q}{4\pi\varepsilon_0 r}\sum_{n=0}^{\infty}P_n(\cos\theta)\left(\frac{a}{r}\right)^n$ valid for $r>a$.

3. For $r<a$, factor $\frac{1}{a}$ instead $\Rightarrow \Psi(r,\theta)=\frac{q}{4\pi\varepsilon_0 a}\frac{1}{\left(1-2\frac{r}{a}\cos\theta+\frac{r^2}{a^2}\right)^{\frac{1}{2}}}=\frac{q}{4\pi\varepsilon_0 a}g\!\left(\cos\theta,\frac{r}{a}\right)$

II. (Continued)

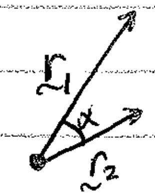
B. Expansion of $\frac{1}{|\underset{\sim}{r_1}-\underset{\sim}{r_2}|}$

1. Setting $\frac{q}{4\pi\varepsilon_0} = 1$, $\psi = \frac{1}{|r_1 - r_2|}$ just depend on separation distance.

a. In fact, alignment with z-axis is unnecessary.

2. Thus

$$\boxed{\frac{1}{|\underset{\sim}{r_1}-\underset{\sim}{r_2}|} = \frac{1}{r_>}\sum_{n=0}^{\infty}\left(\frac{r_>}{r_<}\right)^n P_n(\cos\alpha)}$$



where $r_> = \max(r_1, r_2)$

$r_< = \min(r_1, r_2)$

a. Converges everywhere except $\underset{\sim}{r_1} = \underset{\sim}{r_2}$.

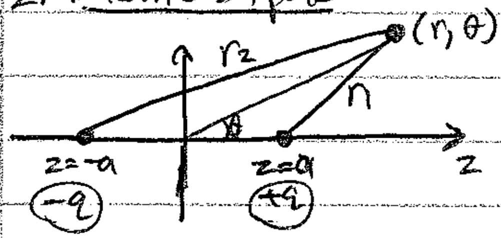
C. Multipole Expansion

1. For $r > a$, $\psi(r,\theta) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r} + \frac{a\cos\theta}{r^2} + \frac{a^2(\frac{3}{2}\cos^2\theta - 1)}{r^3} + \ldots\right]$

Potential if q at $r=0$

First correction (dipole) due to actual position

2. Electric Dipole



a. Monopole term cancels (Net charge zero)

b. $\psi(r,\theta) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

c. Using expansion for

$r_1 = (a^2 - 2ar\cos\theta + r^2)^{\frac{1}{2}}$

$r_2 = (a^2 - 2ar\underbrace{\cos[\pi-\theta]}_{=-\cos\theta} + r^2)^{\frac{1}{2}}$

we obtain

$$\psi(r,\theta) = \frac{q}{4\pi\varepsilon_0 r}\left[\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n - \sum_{n=0}^{\infty} P_n(\cos\theta)\left(-\frac{a}{r}\right)^n\right]$$

d. Even terms cancel, so $\psi(r,\theta) = \frac{2q}{4\pi\varepsilon_0 r}\left\{\frac{a}{r}P_1(\cos\theta) + \frac{a^3}{r^3}P_3(\cos\theta) + \ldots\right\}$

e. Def: Dipole Moment $\mu \equiv 2qa$

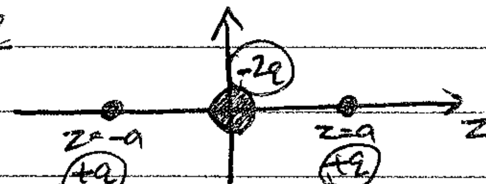
f. Take limit $a \to 0$ while keeping μ = constant, higher-order terms drop.

Point Dipole:

$$\boxed{\Psi(r,\theta) = \frac{\mu}{4\pi\varepsilon_0}\,\frac{P_1(\cos\theta)}{r^2}} \leftarrow \Psi \propto r^{-2}$$

3. Electric Quadrupole

a. Similarly



$\mu_2 = 2qa^2$

Point Quadrupole:

$$\boxed{\Psi(r,\theta) = \frac{\mu_2}{4\pi\varepsilon_0}\,\frac{P_2(\cos\theta)}{r^3}} \leftarrow \Psi \propto r^{-3}$$

4. Multipole Expansion

a. For the 2^n-pole, $\Psi \propto \frac{P_n(\cos\theta)}{r^{n+1}}$ and $\mu_n = \sum_{i=1}^{n} q_i (a_i)^n$

b. For a general distribution of point charges q_i at r_i (on z-axis),

$$\boxed{\Psi(r,\theta) = \frac{1}{4\pi\varepsilon_0 r}\sum_n \frac{\mu_n P_n(\cos\theta)}{r^n}}$$ where $\mu_n = \sum_{i=1}^{N} q_i (r_i)^n$

Axisymmetric (m=0) Multipole Expansion

5. General Comments

a. Multipole expansions can also be used for continuous charge distributions,

$$\mu_n = \int dr\, \rho(r)\, r^n$$

b. In 3D, higher-order moments have more terms ($x^2, xy, y^2, yz, z^2, xz$)

c. Applies to any inverse-square force ⇒ gravity, etc.

III. Associated Legendre Equation

A. Equation

1. $$\boxed{(1-x^2)P''(x) - 2xP'(x) + \left[\lambda - \frac{m^2}{1-x^2}\right]P(x) = 0}$$

2. Taking $P(x) = (1-x^2)^{m/2}\mathcal{P}(x)$, we obtain

$$\boxed{(1-x^2)\mathcal{P}''(x) - 2x(m+1)\mathcal{P}'(x) + [\lambda - m(m+1)]\mathcal{P}(x) = 0}$$

3. Can be solved using Frobenius Method $\sim \sum_j a_j x^{k+j}$
   a. Indicial equation: Solutions $\boxed{k=0}$, $k=1$
   b. Recurrence: $a_{j+2} = a_j\left[\frac{j^2+(2m+1)j-\lambda+m(m+1)}{(j+1)(j+2)}\right]$
   c. To avoid divergence at $x=\mp 1$, must choose λ to terminate series.
   $\Rightarrow$ For $\lambda = \ell(\ell+1)$, series terminate when $j = \ell - m > 0$
   d. Thus ℓ must be at least as large as m and of same parity

4. Def: Associated Legendre Function $\boxed{P_\ell^m(x) = (1-x^2)^{m/2}\mathcal{P}_\ell^m(x)}$
   a. $\mathcal{P}_\ell^m(x)$ is polynomial of degree $\ell - m$

5. Explicit Formula for $\mathcal{P}_\ell^m(x)$:
   a. Can be obtained by repeated differentiation of Legendre Equation
   Leibniz's Formula: $\frac{d^m}{dx^m}[A(x)B(x)] = \sum_{s=0}^m \binom{m}{s}\frac{d^{m-s}A(x)}{dx^{m-s}}\frac{d^s B(x)}{dx^s}$
   b. Applied to regular Legendre Eq yields
   $$(1-x^2)u'' - 2x(m+1)u' + [\ell(\ell+1) - m(m+1)]u = 0$$ Equation for $\mathcal{P}_\ell^m(x)$
   where $u = \frac{d^m}{dx^m}P_\ell(x)$
   c. Thus
   $$\boxed{P_\ell^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(m)}$$

d. When $m=0$, we omit superscript $m \Rightarrow P_\ell^0 \equiv P_\ell$

6. Condition on ℓ & m:

a. For each ℓ, solutions with integer m $0 \le m \le \ell$.

7. Negative Values of m:

a. Only m^2 in Associated Legendre Equation

b. Rodrigues formula for $P_\ell(x)$ can be used to obtain

$$\boxed{P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)}$$

B. Generating Function and Recurrence Formulas

1. $$\boxed{g_m(x,t) = \frac{(-1)^m (2m-1)!!}{(1-2xt+t^2)^{m+\frac{1}{2}}} = \sum_{s=0}^{\infty} P_{s+m}^m(x) t^s}$$

2. $\frac{\partial g_m(x,t)}{\partial t}$ yields $\boxed{(\ell-m+1) P_{\ell+1}^m - (2\ell+1) x P_\ell^m + (\ell+m) P_{\ell-1}^m = 0}$

a. Connects $P_{\ell-1}^m, P_\ell^m, P_{\ell+1}^m$

3. We also note that $(1-2xt+t^2) g_{m+1}(x,t) = -(2m+1) g_m(x,t)$, so

$$\boxed{P_{\ell+1}^{m+1} - 2x P_\ell^{m+1} + P_{\ell-1}^{m+1} = -(2m+1) P_\ell^m}$$

a. Connects $P_{\ell+1}^{m+1}, P_\ell^{m+1}, P_{\ell-1}^{m+1}$ to P_ℓ^m

4. With two indices ℓ & m and possible x-derivatives of $P_\ell^m(x)$, there are a wide variety of recurrence & derivative formulas! (See text for numerous examples)

a. Can construct all $P_\ell(x)$ from $P_0(x)$ & $P_1(x)$ using Legendre recurrence.

b. Can construct all $P_\ell^m(x)$ from $P_\ell^0(x) = P_\ell(x)$ using these associated Legendre recurrence relations. (All $P_\ell^m = 0$ for $m > \ell$!)