Hawes ①

Lecture #13: Associated Legendre Functions, Spherical Harmonics and Second Kind

I. Associated Legendre Functions

A. Parity

1. $$P_\ell^m(-x) = (-1)^{\ell+m} P_\ell^m(x) \quad \text{(Parity of } \ell+m\text{)}$$

2. Values at endpoints $x = \pm 1$:

a. Since $P_\ell^m(x) = (-1)^m \underbrace{(1-x^2)^{\frac{m}{2}}} \frac{d^m}{dx^m} P_\ell(x)$,

$$P_\ell^m(\pm 1) = 0 \text{ for } m \neq 0$$

b. For $m=0$, $P_\ell^0(x) = P_\ell(x)$, so we recover $P_\ell(1) = 1$, $P_\ell(-1) = (-1)^\ell$

3. Value at $P_\ell^m(0)$:

$$P_\ell^m(0) = \begin{cases} (-1)^{(\ell+m)/2} \dfrac{(\ell+m-1)!!}{(\ell-m)!!} & \ell+m \text{ even} \\ 0 & \ell+m \text{ odd} \end{cases}$$

Recall: $(2n)!! = 2^n n!$ and $(2n-1)!! = \frac{(2n)!}{2^n n!}$

B. Orthogonality

1. For each m, all $P_\ell^m(x)$ for $\ell \geq |m|$ are orthogonal!

a. NOTE: Recall for each ℓ, there are $2\ell+1$ possible m values, $-\ell \leq m \leq \ell$.

2. Orthogonality Relation of ℓ & ℓ':

$$\boxed{\int_0^\pi P_p^m(\cos\theta) P_q^m(\cos\theta)\sin\theta\, d\theta = \int_{-1}^{1} P_p^m(x) P_q^m(x)\, dx = \frac{2}{2p+1} \frac{(p+m)!}{(p-m)!} \delta_{pq}}$$

a. In terms of $\mathcal{P}_\ell^m(x)$, where $P_\ell^m(x) = (1-x^2)^{\frac{m}{2}} \mathcal{P}_\ell^m(x)$,

$$\int_{-1}^{1} \mathcal{P}_p^m(x) \mathcal{P}_q^m(x) (1-x^2)^m dx = \frac{2}{2p+1} \frac{(p+m)!}{(p-m)!} \delta_{pq}$$

b. $\Rightarrow$ polynomials $\mathcal{P}_\ell^m(x)$ are orthogonal with weight $(1-x^2)^m$

c. In contrast, $P_\ell^m(x)$ are orthogonal with unit weight.

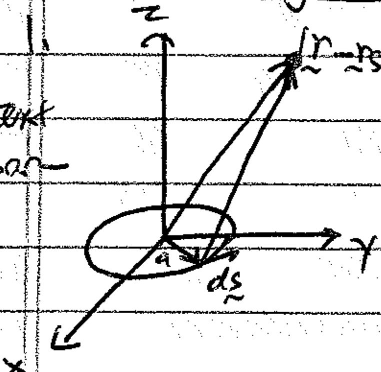
3. Orthogonality Relation of m & m'

a. $$\int_{-1}^{1} P_\ell^m(x)\, P_\ell^n(x)\, (1-x^2)^{-1}\, dx = \frac{(\ell+m)!}{m(\ell-m)!}\delta_{mn}$$

b. Not really very useful: $\Phi(\phi) = a_m e^{\pm im\phi}$ is already orthogonal in m, so we don't generally need to deal with the orthogonality of $P_\ell^m(\cos\theta)$ with respect to m!

C. Ex: Magnetic Field from a Circular Current Loop

NOTE: ~~Figure in text~~ is very poor — loop is not tilted!

1. 

2. $$d\underset{\sim}{A}(\underset{\sim}{r}) = \frac{\mu_0}{4\pi}\frac{I\,ds}{|\underset{\sim}{r}-\underset{\sim}{r_s}|}$$

a. By symmetry, $\underset{\sim}{A} = A_\phi(r,\theta)\,\hat{\underset{\sim}{e}}_\phi$

i) Axisymmetry $\Rightarrow$ cannot depend on ϕ.

ii) $d\underset{\sim}{A}$ is in direction of $d\underset{\sim}{s}$ (in plane of constant z).

iii) In spherical coord's, $\nabla\cdot\underset{\sim}{A} = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(r^2A_r) + r\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + r\frac{\partial A_\phi}{\partial\phi}\right]$

If $A_r \neq 0$ or $A_\theta \neq 0$, then $\nabla\cdot\underset{\sim}{A}$ would diverge at $r=0$.

3. Ampere-Maxwell Law: $\nabla\times\underset{\sim}{B} = \mu_0 \underbrace{\cancel{\underset{\sim}{J}}}_{\text{no current except on loop}}{}^{0} + \epsilon_0\mu_0 \underbrace{\cancel{\frac{\partial \underset{\sim}{E}}{\partial t}}}_{\text{static}}{}^{0} \Rightarrow \nabla\times\underset{\sim}{B} = 0$

4. a. $\underset{\sim}{B} = \nabla\times\underset{\sim}{A}$

b. $\nabla\times\left[\nabla\times A_\phi(r,\theta)\,\hat{\underset{\sim}{e}}_\phi\right] = 0$

$$\Rightarrow \boxed{\frac{\partial^2 A_\phi}{\partial r^2} + \frac{2}{r}\frac{\partial A_\phi}{\partial r} + \frac{1}{r^3\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_\phi}{\partial\theta}\right) - \frac{1}{r^3\sin^2\theta}A_\phi = 0}$$

5. By Usual Separation of Variables, $A_\phi(r,\theta) = R(r)\,\Theta(\theta)$,

a. $$r^2 \frac{d^2R}{dr^2} + 2\frac{dR}{dr} - \ell(\ell+1) R = 0$$

b. $$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\Theta - \frac{\Theta}{\sin^2\theta} = 0$$ Associated Legendre Eq with $\underline{m=1}$

c. General Solution: $$A_\phi(r,\theta) = \sum_{\ell=0}^{\infty}\left[c_\ell\left(\frac{a}{r}\right)^{\ell+1} + \cancel{d_\ell\left(\frac{r}{a}\right)^{\ell}}^{\,0}\right] P_\ell^1(\cos\theta)$$

want solution for $r > a$, so $d_\ell = 0$.

6. Solution for Components of $\underset{\sim}{B}$:

a. $$B_r(r,\theta) = (\nabla \times A_\phi \hat{\underset{\sim}{e}}_\phi)_r = \frac{\cot\theta}{r} A_\phi + \frac{1}{r}\frac{\partial A_\phi}{\partial\theta}$$

b. $$B_\theta(r,\theta) = (\nabla \times A_\phi \hat{\underset{\sim}{e}}_\phi)_\theta = -\frac{1}{r}\frac{\partial(rA_\phi)}{\partial r}$$

7. To evaluate θ derivative, use derivative relation

$$(1-x^2)^{\frac{1}{2}}\left[P_\ell^m(x)\right]' = (\ell+m)(\ell-m+1)P_\ell^{m-1} + \frac{mx}{(1-x^2)^{\frac{1}{2}}}P_\ell^m(x)$$

with $m=1$ and $x = \cos\theta$ to obtain

$$\frac{dP_\ell^1(\cos\theta)}{d\theta} = -\sin\theta\frac{dP_\ell^1(\cos\theta)}{d(\cos\theta)} = -\ell(\ell+1)P_\ell(\cos\theta) - \underbrace{\cot\theta\, P_\ell^1(\cos\theta)}_{\text{Cancels other } \cot\theta \text{ term!}}$$

8. Thus

a. $$\boxed{B_r(r,\theta) = \frac{-1}{r}\sum_{\ell=1}^{\infty}\ell(\ell+1)c_\ell\left(\frac{a}{r}\right)^{\ell+1}P_\ell(\cos\theta)}$$

b. $$\boxed{B_\theta(r,\theta) = \frac{1}{r}\sum_{\ell=1}^{\infty}\ell\, c_\ell\left(\frac{a}{r}\right)^{\ell+1}P_\ell^1(\cos\theta)}$$

9. Now, we must determine coefficients c_ℓ by matching solution

a. Choose polar axis, where we may calculate $B_r = B_z$.

b. Along $\theta = 0$,

$$B_r(z,0) = -\frac{a^2}{z^3}\sum_{s=0}^{\infty}(s+1)(s+2)c_{s+1}\left(\frac{a}{z}\right)^s$$

I. C. (Continued) Howes (4)

10. Along polar axis, Biot-Savart Law $d\mathbf{B} = \frac{\mu_0}{4\pi} I \frac{d\mathbf{s} \times \hat{\mathbf{r}}}{r^2}$ may be integrated to yield

$$B_z = \frac{\mu_0 I a^2}{2z^3}\left(1+\frac{a^2}{z^2}\right)^{-\frac{3}{2}} = \frac{\mu_0 I a^2}{2z^3}\sum_{s=0}^{\infty}(-1)^s \frac{(2s+1)!!}{(2s)!!}\left(\frac{a}{z}\right)^{2s}$$

11. We thereby find

$$\boxed{\begin{aligned} c_{2s} &= 0 \\ c_{2s+1} &= \frac{\mu_0 I}{2}(-1)^{s+1}\frac{(2s-1)!!}{(2s+2)!!} \end{aligned}}$$

II. Spherical Harmonics

A. Generalized Angular (θ, ϕ) Solutions

1. In spherically symmetric problems, $\Theta(\theta)\Phi(\phi)$ are always the same for Laplace, Helmholtz, and Schrödinger equations.

2. Φ Solution: For integer m,

a. $$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \quad \text{or} \quad \Phi_m(\phi) = \frac{1}{\sqrt{\pi}}\begin{cases} 1/\sqrt{2} & m=0 \\ \cos m\phi & m>0 \\ \sin|m|\phi & m<0 \end{cases}$$

b. $$\int_0^{2\pi}\left[\Phi_{m'}(\phi)\right]^* \Phi_m(\phi)\, d\phi = \delta_{mm'} \quad \text{Orthogonality}$$

3. Θ Solution:

a. $$\Theta_{\ell m}(\cos\theta) = \sqrt{\frac{2\ell+1}{2}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta) \quad \text{for } -\ell \le m \le \ell$$

b. Orthogonality $$\int_0^{\pi}\left[\Theta_{\ell m}(\cos\theta)\right]^* \Theta_{\ell' m}(\cos\theta)\sin\theta\, d\theta = \delta_{\ell\ell'}$$

Same value of m!

II. A. (Continued) Howes (5)

4. Def: Spherical Harmonics

a. $$Y_\ell^m(\theta,\phi) \equiv \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\phi}$$

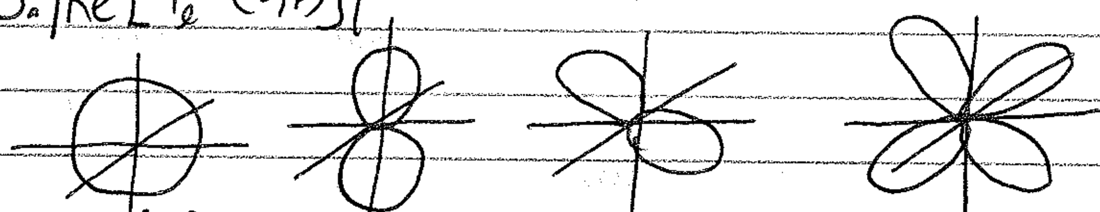
b. $$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \left[Y_\ell^m(\theta,\phi)\right]^* Y_{\ell'}^{m'}(\theta,\phi) = \delta_{\ell\ell'}\delta_{mm'}$$

c. Normalized Solutions of Sturm-Liouville problem.

$\Rightarrow$ Complete set of orthonormal eigenfunctions over spherical surface

5. $|Re[Y_\ell^m(\theta,\phi)]|^2$

a.



$m=0, \ell=0$ $\quad$ $m=0, \ell=1$ $\quad$ $m=1, \ell=1$ $\quad$ $m=1, \ell=2$, ...

b. $\frac{1}{\sqrt{4\pi}}$ $\quad$ $\sqrt{\frac{3}{4\pi}}\cos\theta$ $\quad$ $-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$ $\quad$ $-\sqrt{\frac{5}{24\pi}}\, 3\sin\theta\cos\theta e^{i\phi}$

6. Cartesian Representation: a. $\cos\theta = \frac{z}{r}$

b. $\sin\theta e^{\pm i\phi} = \frac{x}{r} \pm i\frac{y}{r}$

B. General Solutions:

1. Laplace Eq: $\nabla^2\psi = 0$

$$\boxed{\psi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(a_{\ell m} r^{\ell} + b_{\ell m} r^{-\ell-1}\right) Y_\ell^m(\theta,\phi)}$$

2. Helmholtz Eq: $\nabla^2\psi + k^2\psi = 0$

$$\boxed{\psi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[a_{\ell m} j_\ell(kr) + b_{\ell m} y_\ell(kr)\right] Y_\ell^m(\theta,\phi)}$$

Spherical Bessel Functions

C. Laplace Expansion

1. Thm: Any function $f(\theta,\phi)$ on a spherical surface may be expanded in $Y_\ell^m(\theta,\phi)$

a. $$f(\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} C_{\ell m} Y_\ell^m(\theta,\phi)$$

b. where $$C_{\ell m} = \langle Y_\ell^m | f(\theta,\phi) \rangle = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \, Y_\ell^{m*}(\theta,\phi) f(\theta,\phi)$$

2. Ex: Electrostatic Potential within sphere with Potential $V(r_0,\theta,\phi)$

a. General Solution for $r<r_0$ $$V(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} r^\ell Y_\ell^m(\theta,\phi)$$

b. Determine $a_{\ell m}$ on sphere at $r=r_0$:

$$V(r_0,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell m} Y_\ell^m(\theta,\phi) \quad \text{where } c_{\ell m} = a_{\ell m} r_0^\ell$$

and $$\boxed{c_{\ell m} = \langle Y_\ell^m | V(r_0,\theta,\phi) \rangle}$$

c. Thus $$\boxed{V(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell m} \left(\frac{r}{r_0}\right)^\ell Y_\ell^m(\theta,\phi)}$$ with $c_{\ell m}$ above

D. Symmetry

1. NOTE: A spherically symmetric problem can have solutions within spherical symmetry

a. Ex: Earth orbits in a plane, but Sun's gravitational field is spherically symmetric $\Rightarrow$ "initial conditions" determine orbital plane.

2. a. For a given ℓ, $2\ell+1$ mutually orthogonal eigenfunctions, Y_ℓ^m $(-\ell \le m \le \ell)$

b. Any solution (for that ℓ) is a linear combination of these eigenfunctions.

3. Consider angular solution for given l as belonging to a Hilbert space with $(2l+1)$ members.
4. Coordinate system rotations cannot change r dependence, so functions $Y_l^m (-l \leq m \leq l)$ are closed under rotation.

5. <u>Quantum Mechanics:</u>
   a. All solutions with same l have same Energy E and radial functions:
   $\Rightarrow (2l+1)$-fold degeneracy of eigenstates of given l

6. <u>Along polar axis:</u> a. $\theta = 0 \Rightarrow Y_l^m(0,\phi) = \sqrt{\frac{2l+1}{4\pi}}\,\delta_{m0}$

   b. $\theta = \pi \Rightarrow Y_l^m(\pi,\phi) = (-1)^l \sqrt{\frac{2l+1}{4\pi}}\,\delta_{m0}$

7. Note also that one may derive <u>recurrence relations</u> for $Y_l^m(\theta,\phi)$.

III. <u>Legendre Functions of the Second Kind</u>

A. <u>Compute Second, Linearly Independent Solution</u>

1.
$$Y'' - \underbrace{\frac{2x}{1-x^2}}_{=p_1(x)} Y' - \underbrace{\frac{l(l+1)}{1-x^2}}_{=p_2(x)} Y = 0$$

2. Using method of computing $y_2(x)$ from $y_1(x)$ (see Sec 7.6),
$$y_2(x) = y_1(x) \int^x \frac{\exp\left[-\int^{x_2} p_1(x_1)\,dx_1\right]}{[y_1(x_2)]^2}\,dx_2 ,$$
we obtain for $y_1(x) = P_l(x)$,
$$\boxed{Q_l(x) = P_l(x) \int^x \frac{dx}{(1-x^2)[P_l(x)]^2}\,dx}$$

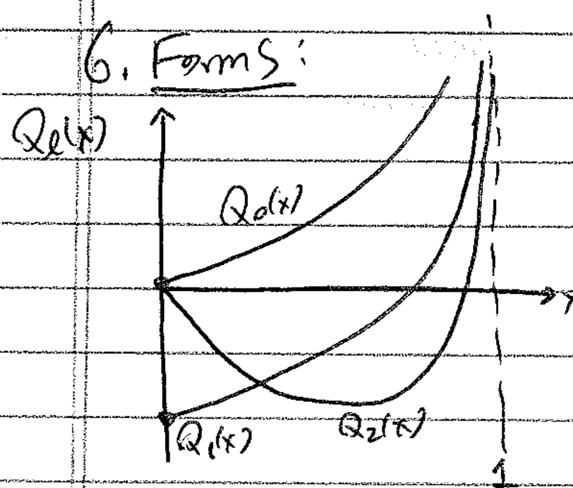
III. A. (Continued) Howes (8)

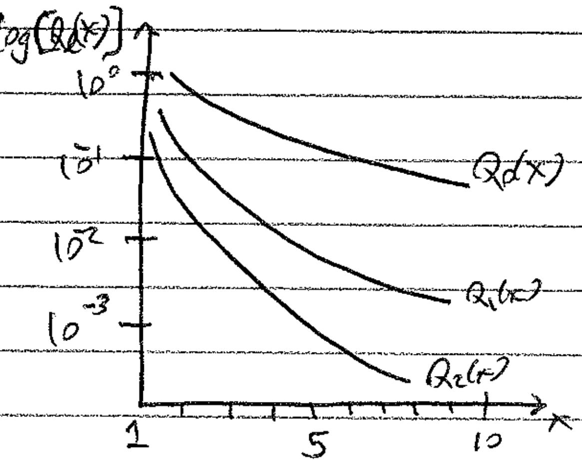
3. Thus,
$$Q_0(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$
$$Q_1(x) = \frac{x}{2} \ln\left(\frac{1+x}{1-x}\right) - 1$$
$$Q_2(x) = \frac{1}{2} P_2(x) \ln\left(\frac{1+x}{1-x}\right) - \frac{3x}{2}$$

4. a. Must obey the same Recurrence Formulas for Legendre Equation.
b. May be used to determine formulas for higher l
$$Q_l(x) = \frac{1}{2} P_l(x) \ln\left(\frac{1+x}{1-x}\right) - \frac{2l-1}{1 \cdot l} P_{l-1}(x) - \frac{2l-5}{3(l-1)} P_{l-3}(x) - \ldots$$

5. $Q_l(x)$ are often used in range $x > 1$ (outside $-1 < x < 1$), so you may replace ln term with $\ln\left(\frac{x+1}{x-1}\right)$

6. Forms:





B. Properties:

1. $Q_l(x)$ is $\begin{cases} \text{odd function} & \text{for even } l \\ \text{even function} & \text{for odd } l \end{cases}$

2. $Q_l(1) = \infty$ for all l

3. $$Q_l(0) = \begin{cases} 0 & \text{for even } l \\ (-1)^{s+1} \dfrac{(2s)!!}{(2s+1)!!} & \text{for odd } l \end{cases}$$