Lecture # 18: Laplace Transforms

I. Laplace Transforms

A. Definition:

1. Consider the transform of a time series $f(t)$.

2. Def: Laplace Transform $\boxed{F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt}$

   a. s may be complex, but we require $Re(s) > 0$ for integral to exist.

3. Existence condition: a. $\int_0^\infty f(t)\, dt$ need not exist!

   b. We require for constants s_0, M and $t_0 \geq 0$

$$\left| e^{-s_0 t} f(t) \right| \leq M \quad \text{for all } t > t_0$$

   Then $\mathcal{L}\{f(t)\}$ will exist for $s > s_0$.

   c. For physical problems, typically a sufficiently large may be chosen to yield a Laplace transform that exists.

4. Laplace Transformation is a linear operation:

$$\mathcal{L}\{a f(t) + b g(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

B. Laplace Transforms of Elementary Functions

1. Ex: $f(t) = 1$ for $t > 0$

   a. $F(s) = \int_0^\infty (1) e^{-st}\, dt = \left[-\frac{e^{-st}}{s} \right]_0^\infty = -\left[\frac{\cancel{e^{-s(\infty)}}^{\,0} - 1}{s} \right] = \boxed{\frac{1}{s} = F(s)}$

2. Ex: $f(t) = e^{kt}$, $t > 0$

   a. $F(s) = \int_0^\infty e^{kt} e^{-st}\, dt = \left[\frac{e^{(k-s)t}}{k-s} \right]_0^\infty = \frac{0 - 1}{k-s} = \boxed{\frac{1}{s-k} = F(s)}$

I. B. (Continued) Howes ③

3. Example: $f(t) = \cosh(kt) = \frac{1}{2}\left[e^{kt} + e^{-kt}\right]$

a. $F(s) = \mathcal{L}\{f(t)\} = \frac{1}{2}\left(\mathcal{L}\{e^{kt}\} + \mathcal{L}\{e^{-kt}\}\right) = \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right)$

$$= \frac{1}{2}\,\frac{(s+\cancel{k})+(s-\cancel{k})}{s^2-k^2} = \frac{\cancel{2}s}{\cancel{2}(s^2-k^2)} = \boxed{\frac{s}{s^2-k^2} = F(s)}$$

4. Example: $f(t) = \cos(kt)$

a. $f(t) = \cos(kt) = \cosh(ikt)$

b. $F(s) = \mathcal{L}\{\cosh(ikt)\} = \frac{s}{s^2-(ik)^2} = \boxed{\frac{s}{s^2+k^2} = F(s)}$

C. Heaviside Step Function

1. With Laplace Transforms, can be used to solve for transient behavior of a system as forcing is switched on at $t = t_0$!

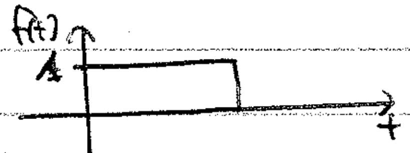
2. Def: Heaviside Step Function

$$U(t-t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases}$$

3. $F(s) = \int_0^\infty U(t-t_0)\, e^{-st}\, dt = \int_{t_0}^\infty e^{-st}\, dt = \left[\frac{-e^{-st}}{s}\right]_{t_0}^{\infty} = -\left[\frac{\cancel{e^{-s(\infty)}}^{\,0} - e^{-st_0}}{s}\right]$

$$= \boxed{\frac{e^{-st_0}}{s} = F(s)}$$

4. Ex: Square Pulse



a. $f(t) = A\left[U(t) - U(t-t_0)\right]$

b. $F(s) = \mathcal{L}\{f(t)\} = A\left[\mathcal{L}\{U(t)\} - \mathcal{L}\{U(t-t_0)\}\right] = A\left[\frac{e^{-s(0)}}{s} - \frac{e^{-st_0}}{s}\right]$

$$\boxed{F(s) = \frac{A}{s}\left(1 - e^{-st_0}\right)}$$

5. Delta Function $f(t) = \delta(t - t_0)$ $t_0 > 0$

a. $\mathcal{L}\{f(t)\} = \int_0^\infty \delta(t-t_0) e^{-st}\, dt = \boxed{e^{-st_0} = F(s)}$

b. Taking the limit $\lim_{t_0 \to 0} F(s) = 1 \Rightarrow \boxed{\mathcal{L}\{\delta(t)\} = 1}$ Impulse Function

D. Inverse Laplace Transforms

1. Computing the inverse Laplace Transform is often the most challenging step when using Laplace Transforms to solve problems.

2. Tables of Laplace Transforms (Table 20.1 in text, p. 1012)

$f(t)$	$F(s)$	Limitation
e^{kt}	$\frac{1}{s-k}$	$s > k$
te^{kt}	$\frac{1}{(s-k)^2}$	$s > k$
$\sin kt$	$\frac{k}{s^2+k^2}$	$s > 0$
$e^{at}\cos kt$	$(s-a)/[(s-a)^2+k^2]$	$s \geq a$
⋮	⋮	⋮

3. Complex Contour Integration using Residue Theorem (later in Sec 20.10)

4. Sometimes a Partial Fraction Expansion is helpful to use Table.

a. $F(s) = \frac{k^2}{s(s^2+k^2)} = \frac{1}{s} - \frac{s}{s^2+k^2}$

b. $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \boxed{1 - \cos kt = f(t)}$

II. Properties of Laplace Transforms

A. Transforms of Derivatives

1. Laplace transforms can be used to convert ODEs into algebraic eq's.

2. Consider $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st}\frac{df(t)}{dt}dt = \left[e^{-st}f(t)\right]_0^\infty - \int_0^\infty (-se^{-st})f(t)dt$

a. Integrate by parts $u = e^{-st}$ $dv = \frac{df}{dt}dt$

$du = -se^{-st}dt$ $v = f$

$$= \left[\cancelto{0}{e^{-s\infty}}f(\infty) - \cancelto{1}{e^{-s0}}f(0)\right] + s\int_0^\infty e^{-st}f(t)dt = sF(s) - f(0)$$

Initial condition in physical space $f(t)|_{t=0}$

b. Thus $\boxed{\mathcal{L}\{f'(t)\} = sF(s) - f(0)}$

3. For higher order derivatives, $\boxed{\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)}$

4. Beware: Coefficients of differential equation must be constant (not dependent on independent variable t) for this method to be applied.

5. Initial Conditions

a. Initial conditions at $t=0$ [ie., $f(0)$, $f'(0)$, etc.] are automatically incorporated into the solution.

b. Powerful method for solving Initial Value Problems.

B. Example: Simple Harmonic Oscillator

1. Mass m on spring of constant k,

a. $m\frac{d^2y(t)}{dt^2} + ky(t) = 0$

b. Initial Conditions: $y(0) = A$, $y'(0) = 0$

2. Laplace Transform Equation:

a. $m\mathcal{L}\left\{\frac{d^2y(t)}{dt^2}\right\}+k\mathcal{L}\left\{y(t)\right\}=ms^2Y(s)-ms\overset{A}{\cancel{y(0)}}-\overset{0}{\cancel{y'(0)}}+kY(s)=0$

b. Solve for $Y(s)$: $Y(s)\left[ms^2+k\right]=msA$

$$Y(s)=A\frac{s}{s^2+k/m}\Rightarrow\boxed{Y(s)=A\frac{s}{s^2+\omega_0^2}}\quad \text{where } \omega_0^2=\frac{k}{m}$$

3. Inverse Laplace Transform: From Table 20.1,

a. $f(t)=\cos kt \iff F(s)=\dfrac{s}{s^2+k^2}$

b. Thus, $\boxed{f(t)=A\cos(\omega_0 t)}$

4. NOTE: Initial conditions are already satisfied:

a. $f(0)=A\cos(0)=A$ ✓

b. $f'(t)=-\omega_0 A\sin\omega_0 t \Rightarrow f'(0)=-\omega_0 A\sin(0)=0$ ✓

C. Ex: Impulsive Force

1. Impulsive Force at $t=0$ of form $F=P\delta(t)$, $P=$ constant

2. $m\dfrac{d^2x(t)}{dt^2}=P\delta(t)$; $x(0)=0$, $x'(0)=0$

3. Laplace Transform equation: $m\mathcal{L}\left\{\frac{d^2x}{dt^2}\right\}=P\mathcal{L}\{\delta(t)\}$

a. $ms^2X(s)-ms\overset{0}{\cancel{x(0)}}-\overset{0}{\cancel{x'(0)}}=P(1)$

4. Solve for $X(s)$: $X(s)=\dfrac{P}{ms^2}$

5. Inverse Transform: $f(t)=t^n \iff F(s)=\dfrac{\Gamma(n+1)}{s^{n+1}}$

a. For $n=1$, $F(s)=\dfrac{\Gamma(2)}{s^2}=\dfrac{1}{s^2}\Rightarrow f(t)=t$

b. Thus $\boxed{x(t)=\dfrac{P}{m}t}$

II. (Continued)

D. More Properties

1. Change of Scale:

Change variable $u = at$, $du = a\,dt$

a. $\mathcal{L}\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt = \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)u} f(u)\,du = \frac{1}{a} F\left(\frac{s}{a}\right)$

b. Thus $\boxed{\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)}$

2. Translation: $s = s' - a$

a. $F(s'-a) = \int_0^\infty e^{-(s'-a)t} f(t)\,dt = \int_0^\infty e^{-s't}\left[e^{at} f(t)\right]dt = \mathcal{L}\{e^{at} f(t)\}$

b. Thus $\boxed{\mathcal{L}\{e^{at} f(t)\} = F(s-a)}$

E. Example: Damped Harmonic Oscillator

1. Consider a mass m on a spring (k) with damping coefficient b:

a. $m \frac{d^2 y(t)}{dt^2} + b \frac{dy(t)}{dt} + k\,y(t) = 0$

b. Again, take initial conditions $y(0) = A$, $y'(0) = 0$.

2. Laplace Transform:

a. $m\left[s^2 Y(s) - s\,\cancelto{A}{y(0)} - \cancelto{0}{y'(0)}\right] + b\left[s\,Y(s) - \cancelto{A}{y(0)}\right] + k\,Y(s) = 0$

b. $Y(s)\left[ms^2 + bs + k\right] - A\left[ms + b\right] = 0$

c. $Y(s) = A \dfrac{s + \frac{b}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}$ As before, $\omega_0^2 \equiv \frac{k}{m}$

3. To put into a form to use a Laplace Transform table, complete the square of the denominator:

a. $$\left(s^2+\frac{b}{m}s+\omega_0^2\right)=\left(s^2+\frac{b}{m}s+\frac{b^2}{4m^2}\right)+\left(\omega_0^2-\frac{b^2}{4m^2}\right)$$

$$=\left(s+\frac{b}{2m}\right)^2+\left(\omega_0^2-\frac{b^2}{4m^2}\right)$$

4. Express numerator in terms of $s+\frac{b}{2m}$, yielding,

$$Y(s)=A\,\frac{\left(s+\frac{b}{2m}\right)+\frac{b}{2m}}{\left(s+\frac{b}{2m}\right)^2+\left(\omega_0^2-\frac{b^2}{4m^2}\right)}$$ Define $\omega_1^2\equiv\left(\omega_0^2-\frac{b^2}{4m^2}\right)$

5. Inverse transform: Thus oscillatory!

a. For the weakly damped case (yielding ω_1 real), we have

$$\omega_0^2>\frac{b^2}{4m^2}\Rightarrow\frac{k}{m}>\frac{b^2}{4m^2}\Rightarrow\boxed{b^2<4km}$$

b. In this case, $f(t)=e^{at}\cos kt \Leftrightarrow F(s)=\frac{s-a}{(s-a)+k^2}$

and $f(t)=e^{at}\sin kt \Leftrightarrow F(s)=\frac{k}{(s-a)+k^2}$

c. Therefore; for $a=\frac{-b}{2m}$ and $k=\omega_1$,

$$\mathcal{L}^{-1}\{Y(s)\}=A\left[e^{\frac{-bt}{2m}}\cos\omega_1 t+\frac{b}{2m\omega_1}e^{\frac{-bt}{2m}}\sin\omega_1 t\right]$$

6. Solution: $$\boxed{y(t)=Ae^{\frac{-bt}{2m}}\left(\cos\omega_1 t+\frac{b}{2m\omega_1}\sin\omega_1 t\right)}$$

a. NOTE: In the limit $b\to 0$, we obtain $\omega_1=\sqrt{\omega_0^2-\frac{b^2}{4m^2}}=\omega_0$, so

$$y(t)=A\cos\omega_0 t$$

agreeing with example in II.B. above.