Lecture #2 Cauchy's Integral Formula, Laurent Expansions, and Singularities

I. Cauchy's Integral Formula

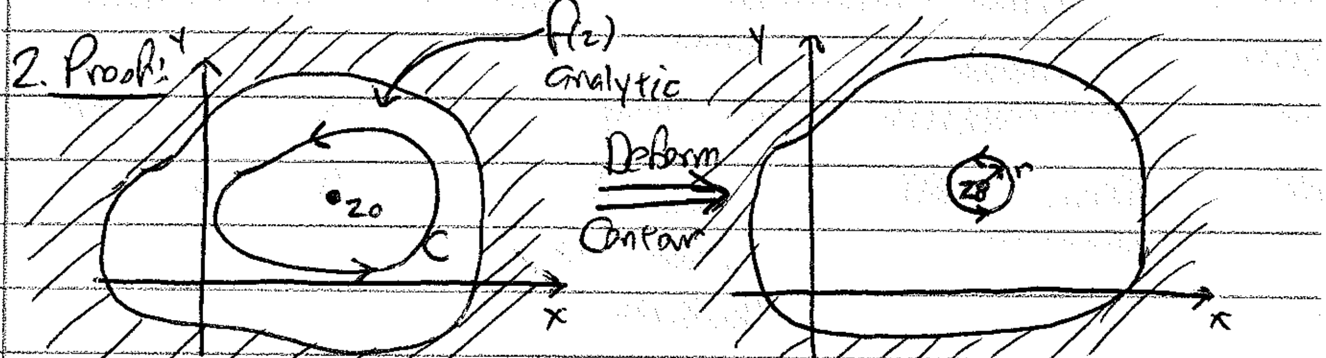
A.

1. Cauchy's Integral Formula

$$\boxed{\frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z-z_0} = \begin{cases} f(z_0) & \text{, if } z_0 \text{ is within } C \\ 0 & \text{, if } z_0 \text{ is not within } C \end{cases}}$$

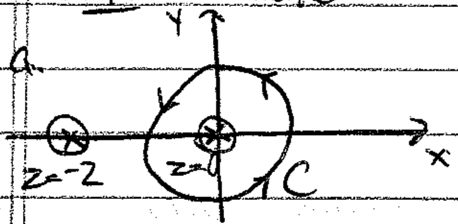
a. Here, $f(z)$ is an analytic function within and on contour C.
b. C is traversed in the counterclockwise direction about z_0
c. Since z_0 is interior to C, $z-z_0 \neq 0$ along integration contour.
d. NOTE: $\frac{f(z)}{z-z_0}$ is not analytic at $z=z_0$ unless $f(z_0)=0$.

2. Proof:



a. Deform contour C over analytic region to circle of radius r about z_0
b. Let $z = z_0 + re^{i\theta}$, $dz = ire^{i\theta}d\theta$
c. $\oint \frac{f(z)}{z-z_0}dz = \oint_0^{2\pi} \frac{f(z_0+re^{i\theta})}{\cancel{re^{i\theta}}} i\cancel{re^{i\theta}}d\theta \underset{\lim_{r\to 0}}{=} i\oint_0^{2\pi} f(z_0)\,d\theta = 2\pi i f(z_0)$
d. NOTE: Since $f(z)$ is analytic and continuous at $z=z_0$,
$\lim_{r\to 0} f(z_0+re^{i\theta}) = f(z_0)$
e. Thus $\frac{1}{2\pi i}\oint \frac{f(z)}{z-z_0}dz = f(z_0)$. ✓

3. Ex: Evaluate $I = \oint_C \frac{dz}{z(z+2)}$ with C an unit circle

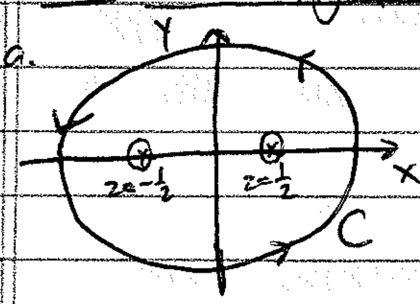
a. 

Not analytic at $z=0$, $z=-2$.

b. May be written $I = \oint \frac{f(z)}{z-z_0} dz$ where $f(z) = \frac{1}{z+2}$ and $z_0 = 0$

c. Thus $I = 2\pi i f(z_0) = 2\pi i \left[\frac{1}{\cancel{z_0}+2}\right] = \boxed{i\pi}$.

4. Ex: Two Singular Factors: $I = \oint_C \frac{dz}{4z^2-1}$ on unit circle C.

a. 

$= \oint \frac{dz}{4(z-\frac{1}{2})(z+\frac{1}{2})}$

b. Later, Residues will make this calculation simple. For now, we can use a partial fraction expansion to use Cauchy's Integral Formula.

$$\frac{1}{4z^2-1} = \frac{1}{4}\left(\frac{1}{z-\frac{1}{2}} - \frac{1}{z+\frac{1}{2}}\right)$$

c. Thus $I = \frac{1}{4}\left\{\oint \frac{dz}{z-\frac{1}{2}} - \oint \frac{dz}{z+\frac{1}{2}}\right\}$ ← in both cases $f(z) = 1$.

d. $I = \frac{1}{4}\left\{2\pi i \overset{1}{\cancel{f}}(\tfrac{1}{2}) - 2\pi i \overset{1}{\cancel{f}}(-\tfrac{1}{2})\right\} = \boxed{0}$.

B. Higher Order Poles

1. Differentiating Cauchy's Integral Formula with respect to z_0 yields,

$$f'(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^2} dz$$

I.B. (Continued)

2. Continuing iteratively,
$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^{n+1}}\,dz}$$

3. NOTE: a. If $f(z)$ is analytic, not only first derivative, but derivatives of all orders, exist.
   b. The derivatives are automatically analytic.

C. Morera's Theorem

> If a function $f(z)$ is continuous in a simply connected region R and $\oint_C f(z)\,dz = 0$ for every closed contour within R, then $f(z)$ is analytic throughout R.

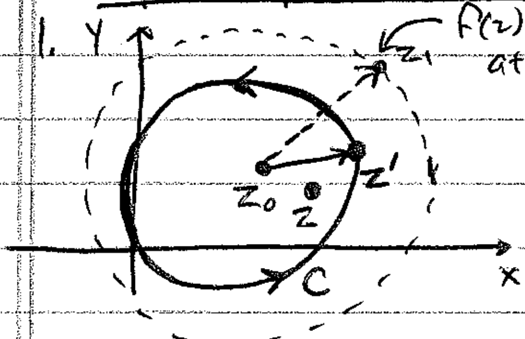
1. A consequence of the proof for Morera's Thm (in text) is that

> The rules for integration of complex functions are same as for real functions

II. Taylor and Laurent Expansions

A. Taylor Expansion for Complex Functions

1.



a. $$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{z'-z} = \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)-(z-z_0)}$$
$$= \frac{1}{2\pi i}\oint \frac{f(z')\,dz'}{(z'-z_0)\left[1-\underbrace{\frac{(z-z_0)}{(z'-z_0)}}_{<1}\right]}$$

2. For complex variable t,
a. $$\frac{1}{1-t} = 1+t+t^2+t^3+\ldots = \sum_{n=0}^{\infty} t^n$$, uniformly convergent for $|t|<1$.

b. So, we may expand [ ] in denominator

II. A. (Continued)

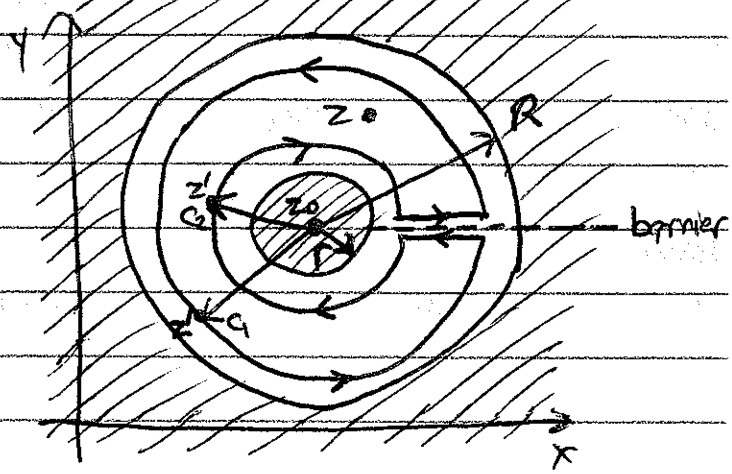
3. $$f(z) = \frac{1}{2\pi i} \oint_C \sum_{n=0}^{\infty} \frac{(z-z_0)^n f(z')dz'}{(z'-z_0)^{n+1}} = \sum_{n=0}^{\infty} (z-z_0)^n \underbrace{\left[\frac{1}{2\pi i} \oint_C \frac{f(z')dz'}{(z'-z_0)^{n+1}}\right]}_{= \frac{f^{(n)}(z_0)}{n!}}$$

4. Thus, $$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n}$$ Taylor Expansion

5. Def: Radius of Convergence, $|z_1 - z_0|$

The Taylor expansion converges within $|z_1 - z_0|$, with a unique expansion of $f(z)$ about a point z_0 within a circle of radius $|z_1 - z_0|$.

B. Laurent Expansion



1. Consider a function $f(z)$ that is analytic in some annular region $r < |z - z_0| < R$ about z_0

2. Over contour $C = C_1 + C_2$ (in simply connected region created by barrier), $$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')dz'}{z'-z} + \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')dz'}{z'-z}$$ (CCW) (CW) (Straight segments Cancel)

3. Using the same approach, writing
$z' - z = (z'-z_0) - (z - z_0)$,
factoring out $(z'-z_0)$ for C_1 and $(z-z_0)$ for C_2, we may obtain an expression that can be written in the form

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{2\pi i} \oint_{C_1} \frac{f(z')dz'}{(z'-z_0)^{n+1}} + \sum_{n=-1}^{-\infty} (z-z_0)^n \frac{1}{2\pi i} \oint_{C_2} (z'-z_0)^{-n-1} f(z')dz'$$

(CCW) now (CCW)

II. B. (Continued)

4. These may be combined to a single series,

Laurent Series

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \qquad a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)^{n+1}}}$$

where the contour C is any CCW contour with $r < |z-z_0| < R$.

5. Negative powers of Laurent Series → always diverges at $z=z_0$, and possibly whole inner region $|z-z_0| \le r$.

6. Ex: Compute Laurent expansion about $z_0=0$ of $f(z) = \frac{1}{z(z-1)}$.

a. Function diverges at $z=0$ and $z=1$, so $r=0$ and $R=1$.
Function is analytic in annular region $0 < |z-z_0| < 1$ (where $z_0=0$)

b. Laurent Expansion:

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} a_n (z)^n} \qquad a_n = \frac{1}{2\pi i}\oint \frac{dz'}{(z')^{n+2}(z'-1)}$$

c. Using $g^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{g(z)\,dz}{(z-z_0)^{n+1}}$ where $g(z) = \frac{1}{z-1}$ and $z_0=0$,

So $a_n = \frac{1}{(n+1)!}\left[\frac{(n+1)!}{2\pi i}\oint \frac{g(z)\,dz}{z^{n+2}}\right] = \frac{1}{(n+1)!}\, g^{(n+1)}(0)$

d. It can be shown, for $g(z) = \frac{1}{z-1}$, that $g^{(n)}(0) = -n!$, so

$$a_n = \frac{-\cancel{(n+1)!}}{\cancel{(n+1)!}} = -1$$

e. $$\boxed{a_n = \begin{cases} -1 & n \ge -1 \\ 0 & n < -1 \end{cases}}$$ ← no singularity for $n<-1$, so $a_n = 0$!

III. Singularities

A. Poles

1. Def: Isolated Singular point: $f(z)$ is not analytic at $z=z_0$ but is analytic at all neighboring points.

2. Laurent expansion about isolated singular point z_0 will have
either a) Finite most negative power, $(z-z_0)^{-n} \rightarrow$ $\boxed{\text{Pole of order } n}$
or b) Negatively infinite powers of $(z-z_0)$. $\boxed{\text{Essential Singularity}}$

3. Pole of order 1 is a simple pole

4. Essential Singularities may be identified by Laurent expansions.
   a. Ex: $$e^{\frac{1}{z}} = 1 + \left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \ldots = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$$
   $\rightarrow$ essential singularity at $z=0$!

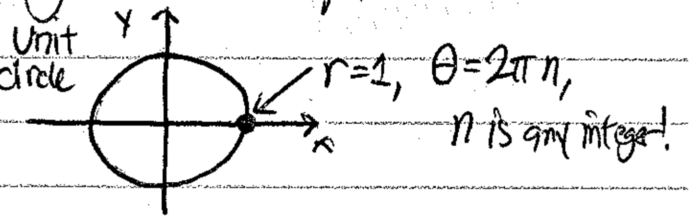
5. Singularities at $z=\infty$:
   a. Behavior of $f(z)$ at $z \rightarrow \infty$ is defined using $f\left(\frac{1}{t}\right)$ as $t \rightarrow 0$.
   b. Ex: $\sin(z)$
      i) $$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$
      ii. $$\sin\left(\frac{1}{t}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\, t^{2n+1}}$$
      iii. Thus, $\sin z$ has an essential singularity at $z=\infty$.
   c. Note: $z = x + iy$.
      i. $|\sin x| \leq 1$ for all real x
      ii. But, $\sin(iy) = i\sinh(y) \rightarrow \infty$ as $y \rightarrow \infty$

B. Making Complex Functions Single-Valued: Branch Points & Branch Cuts

1. Analytic functions must be single-valued, but many complex functions have multiple values.
2. Branch points and branch cuts are used to restrict a complex function to a single-valued region, enabling application of Cauchy's Theorems.

3. Strategy:
a. At point z_0, choose particular value of a multi-valued function.
b. Following along a path in complex plane, value $f(z)$ will change continuously, removing ambiguity if path is open.
c. On a closed path, branch cuts are used to prevent a discontinuity in the value along the closed path.

4. Polar Representation a. $z = re^{i\theta}$ of Multivalued Functions:

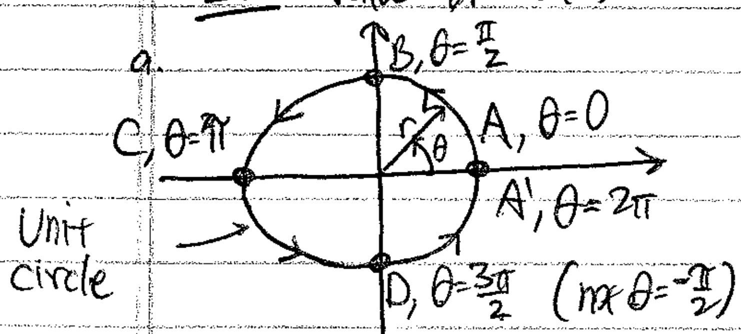


b. $f(z) = z^{\frac{1}{2}} = r^{\frac{1}{2}} e^{i\frac{\theta}{2}}$

c. Thus, $f(z) = (1)^{\frac{1}{2}} e^{i\frac{2\pi n}{2}} = e^{in\pi} = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$ (Two-valued function)

5. Ex: Value of $f(z) = z^{\frac{1}{2}}$ on a Closed Contour

a.



b.

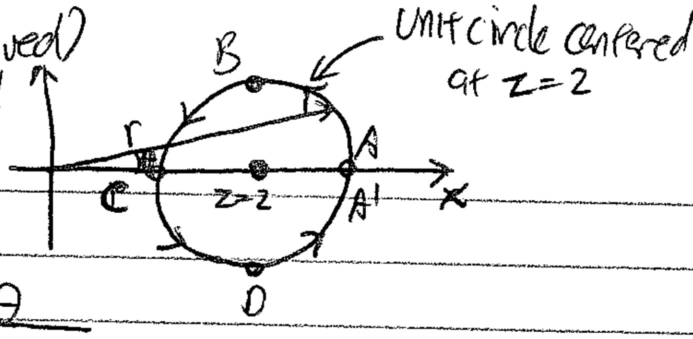
Point	θ	$f(z) = z^{\frac{1}{2}}$
A	0	$e^{0} = 1$
B	$\frac{\pi}{2}$	$e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}$
C	π	$e^{i\frac{\pi}{2}} = +i$
D	$\frac{3\pi}{2}$	$e^{i\frac{3\pi}{4}} = \frac{-1+i}{\sqrt{2}}$
A'	2π	$e^{i\pi} = -1$
A''	4π	$e^{i2\pi} = 1$

Double Valued

c. Two values of $f(z) = z^{\frac{1}{2}}$ at same point in complex plane, A & A'.

III. B. (Continued)

6. Ex: a. $f(z) = z^{\frac{1}{2}}$

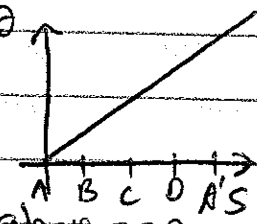


b.

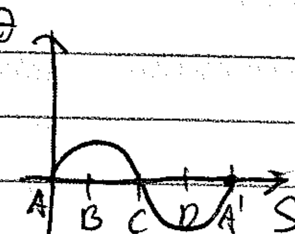
Point	θ
A	0
B	$\sin^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$
C	0
D	$\sin^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$
A'	0

Remains Single-valued

c. Difference: For distance S along contour



C about z=0



C about z=2

7. Key difference between two contours:

a. $f(z) = z^{\frac{1}{2}}$ $\quad f'(z) = \frac{1}{2z^{\frac{1}{2}}} \longleftarrow$ Singular point at z=0!

b. Paths that circle singular point (z=0) lead to ambiguity!

8. a. Def: Branch Point: Singular point about which a contour path will lead to ambiguity of a multi-valued function.

b. Def: Order of a Branch Point: Number of times to encircle branch point before returning to original value

i. Ex: For $f(z) = z^{\frac{1}{2}}$, z=0 is branch point of order 2.

9. Def: Branch Cut: A line from a branch point drawn to either infinity or another finite branch point that a contour path may not cross (to maintain single-valuedness).

a. NOTE: Precise path may be freely chosen, but endpoints are critical.

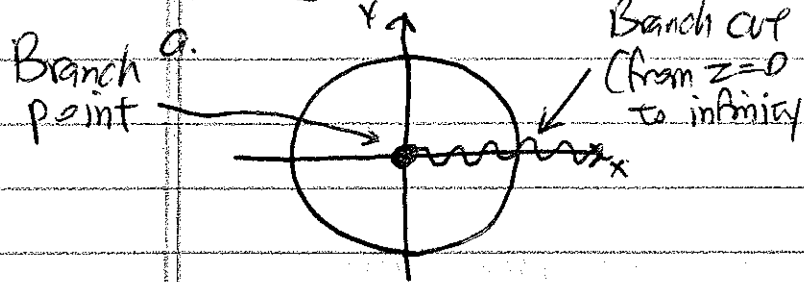
10. <u>Ex</u>: $f(z) = z^{\frac{1}{2}}$

<u>Two single-valued branches</u>

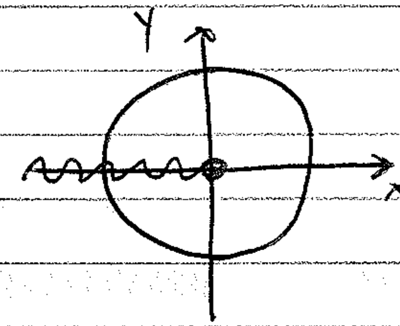
$0 \leq \theta < 2\pi$

$2\pi < \theta \leq 4\pi$

a.



b. Alternatively, we could choose

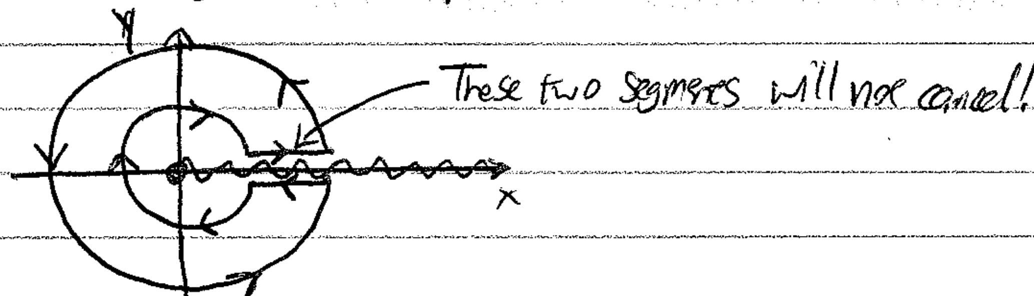


$-\pi < \theta \leq \pi$

$\pi < \theta \leq 3\pi$

11. <u>NOTE</u>: A function $f(z)$ with a branch point and branch cut is <u>not continuous</u> across the branch cut.

a. Thus, line segments on opposite sides will <u>not cancel</u>.



b. Real boundaries of analyticity (unlike artificial barriers introduced to handle multiply connected regions).

12. <u>BOTTOM LINE</u>: a. By drawing appropriate branch cuts, we restrict multi-valued functions to regions of single value, allowing regions bounded by branch cuts to be <u>analytic</u>.

b. Thus, we may apply Cauchy's theorems!