# Lecture #21 Methods for Solving Integral Equations

## I. Special Methods for Solving Integral Equations

### A. Standard Integral Transform

1. If the kernel $K(x,t)$ matches that of a known integral transform, you can use the inverse formula to obtain the solution:

| Method | $K(x,t)$ | Solution |
|---|---|---|
| a. Fourier, $\hat{f}(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixt} f(t)\,dt$ | $e^{ixt}$ | $f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixt}\hat{f}(x)\,dx$ |
| b. Laplace, $\hat{f}(x)=\int_0^{\infty} e^{-xt} f(t)\,dt$ | $e^{-xt}$ | $f(t)=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{xt}\hat{f}(x)\,dx$ |
| c. Mellin, $\hat{f}(x)=\int_0^{\infty} t^{x-1} f(t)\,dt$ | $t^{x-1}$ | $f(t)=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} t^{-x}\hat{f}(x)\,dx$ |
| d. Hankel, $\hat{f}(x)=\int_0^{\infty} t f(t) J_\nu(xt)\,dt$ | $tJ_\nu(xt)$ | $f(t)=\int_0^{\infty} x\hat{f}(x) J_\nu(xt)\,dx$ |

2. One may also invoke the Convolution Theorem to extend these limited results to more general forms (see examples in text).

### B. Generating-Function Method

1. One can use forms of a generating function to evaluate the integral using the orthogonality condition.

2. Consider
$$f(x)=\int_{-1}^{1}\frac{\phi(t)}{(1-2xt+x^2)^{\frac{1}{2}}}\,dt \qquad -1\le x\le 1$$

a. Recall the Legendre polynomial generating function
$$g(x,t)=(1-2xt+x^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty} P_n(t)x^n$$

b. Since $f(x) = \int_{-1}^{1} g(x,t)\,\phi(t)\,dt = \sum_{n=0}^{\infty} \int_{-1}^{1} \phi(t) P_n(t) x^n\,dt$ limits must match

c. Next, we expand the unknown function $\phi(t)$ as a Legendre series $\phi(t) = \sum_{m=0}^{\infty} a_m P_m(t)$

d. Thus, $f(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_m x^n \underbrace{\int_{-1}^{1} P_n(t) P_m(t)\,dt}_{=\frac{2\delta_{nm}}{2n+1} \text{ by Orthogonality}} = \sum_{n=0}^{\infty} \frac{2a_n x^n}{2n+1}$

e. Expanding $f(x)$ as a Taylor expansion about $x=0$, one may solve for $a_n$,

$$a_n = \frac{2n+1}{2} \frac{f^{(n)}(0)}{n!} \Rightarrow \boxed{\phi(x) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \frac{f^{(n)}(0)}{n!} P_n(x)}$$

3. NOTE: Expansion of the unknown function in terms of special functions is always available if the integration interval is appropriate.

C. Separable Kernel

1. If the Kernel $K(x,t)$ can be expressed as a sum of separable functions, $K(x,t) = \sum_{j=1}^{n} M_j(x) N_j(t)$ where $n$ is finite, the problem can be related to eigenvalue equations.

2. For Example, $K(x,t) = \cos(t-x) = \cos t \cos x + \sin t \sin x$

3. Relation to Matrix Eigenvalue Equation

a. Fredholm Eq. of the Second Kind with $K(x,t) = \sum_{j=1}^{n} M_j(x) N_j(t)$,

$$\phi(x) = f(x) + \lambda \sum_{j=1}^{n} M_j(x) \underbrace{\int_a^b N_j(t)\,\phi(t)\,dt}_{\text{Integral is a constant, } c_j}$$

b. Thus $\phi(x) = f(x) + \lambda \sum_{j=1}^{n} c_j M_j(x)$ Once $c_j$ are determined, problem is solved.

c. Convert to Matrix Equation:

Multiply above equation by $N_i(x)$ and integrate $\int_a^b dx$

$$c_i = b_i + \lambda \sum_{j=1}^{n} a_{ij} c_j$$

where $b_i = \int_a^b N_i(x) \underset{\text{Known}}{f(x)}\,dx \qquad a_{ij} = \int_a^b N_i(x) M_j(x)\,dx$

d. Write as a matrix equation: $\underset{\sim}{c} = \underset{\sim}{b} + \lambda \underset{\approx}{A}\,\underset{\sim}{c}$

e. Solve for $\underset{\sim}{c}$: $\boxed{\underset{\sim}{c} = (\underset{\approx}{1} - \lambda \underset{\approx}{A})^{-1} \cdot \underset{\sim}{b}}$

NOTE: For a homogeneous integral equation $[f(x) = 0]$, $\underset{\sim}{b} = 0$.
Thus, eigenvalues are solutions of $|\underset{\approx}{1} - \lambda \underset{\approx}{A}| = 0$.

4. Ex: Homogeneous Fredholm Equation

a. $\phi(x) = \lambda \int_{-1}^{1} (t + x)\,\phi(t)\,dt \Rightarrow$ $M_1(x) = 1$, $M_2(x) = x$; $N_1(t) = t$, $N_2(t) = 1$

b. In this case, $a_{ij} = \int_{-1}^{1} N_i(x) M_j(x)\,dx \Rightarrow \underset{\approx}{A} = \begin{pmatrix} 0 & \frac{2}{3} \\ 2 & 0 \end{pmatrix}$

c. Therefore $|\underset{\approx}{1} - \lambda \underset{\approx}{A}| = \begin{vmatrix} 1 & -\frac{2\lambda}{3} \\ -2\lambda & 1 \end{vmatrix} = 1 - \frac{4\lambda^2}{3} = 0$

d. <u>Solutions</u>: $\lambda = \pm \frac{\sqrt{3}}{2}$

e. Thus, $(\underset{\approx}{1} - \lambda \underset{\approx}{A}) \cdot \underset{\sim}{c} = 0 \Rightarrow c_1 \mp \frac{c_2}{\sqrt{3}} = 0$

f. Choosing $c_1 = 1$, we obtain $\phi_1(x) = \frac{\sqrt{3}}{2}(1 + \sqrt{3}x)$ for $\lambda = \frac{\sqrt{3}}{2}$

and $\phi_2(x) = -\frac{\sqrt{3}}{2}(1 - \sqrt{3}x)$ for $\lambda = \frac{-\sqrt{3}}{2}$

# II. Solution by Neumann Series

## A. A More General Approach: Successive Approximations

1. The techniques of Section I. above require quite special circumstances. Here we describe a more general technique for solving integral equations by Neumann, Liouville, & Volterra.

2. Develop $\phi(x)$ as a power series in $\lambda$ (a constant).
   $\Rightarrow$ Method of <u>successive approximations</u>

3. <u>Fredholm equation of the Second Kind</u>

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\, \phi(t)\, dt$$

4. For $f(x) \neq 0$ and $\lambda$ "small",

a. <u>First Approximation</u>: $\lambda \to 0 \Rightarrow \phi(x) \simeq \phi_0(x) = f(x)$

b. If you can make a better first guess than $f(x)$, use it!

5. a. <u>Second Approximation</u>: $\phi_1(x) = f(x) + \lambda \int_a^b K(x,t)\, \phi_0(t)\, dt$

b. <u>Third Approximation</u>: $\phi_2(x) = f(x) + \lambda \int_a^b K(x,t)\, \phi_1(t)\, dt$

6. NOTE: Substituting in $\phi_1(x)$ and $\phi_0(x)$,

$$\phi_2(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\,dt + \lambda^2 \int_a^b dt_1 \int_a^b dt_2\, K(x,t_1) K(t_1,t_2) f(t_2)\,dt_2\,dt_1$$

etc.

7. In general, $\phi_n(x) = \sum_{i=0}^{n} \lambda^i U_i(x)$

where $u_n(x) = \int_a^b K(x,t_1)\left[\int_a^b K(t_1,t_2)\left\{\int_a^b K(t_2,t_3) \ldots K(t_{n-1},t_n) f(t_n)\,dt_n\right\} \ldots dt_2\right] dt_1$

8. Solution $\phi(x) = \lim_{n\to\infty} \phi_n(x) = \lim_{n\to\infty} \sum_{i=0}^{n} \lambda^i U_i(x)$

$\Rightarrow$ Correct solution if infinite series converges.

B. Ex: Neumann Series Solution

1. $\phi(x) = x + \frac{1}{2}\int_{-1}^{1} (t-x)\,\phi(t)\,dt$

2. $\phi_0(x) = x$

3. $\phi_1(x) = x + \frac{1}{2}\int_{-1}^{1} (t-x)\,t\,dt = x + \frac{1}{2}\left[\frac{t^3}{3} - \frac{xt^2}{2}\right]_{-1}^{1} = x + \frac{1}{3}$

4. $\phi_2(x) = x + \frac{1}{2}\int_{-1}^{1} (t-x)\left[t + \frac{1}{3}\right] dt = x + \frac{1}{3} - \frac{x}{3}$

5. By induction, $\phi_{2n}(x) = x + \sum_{s=1}^{n} (-1)^{s-1} 3^{-s} - x \sum_{s=1}^{n} (-1)^{s-1} 3^{-s}$

6. a. $\lim_{n\to\infty} \sum_{s=1}^{n} (-1)^{s-1} 3^{-s} = -\sum_{s=1}^{\infty} \left(\frac{-1}{3}\right)^s + 1 = 1 - \left(\frac{1}{1-\left(\frac{-1}{3}\right)}\right) = 1 - \frac{3}{4} = \frac{1}{4}$

b. Thus $\phi_{2n}(x) = x + \frac{1}{4} - x\left(\frac{1}{4}\right) = \boxed{\frac{3}{4}x + \frac{1}{4} = \phi_{2n}(x)}$

7. CHECK: $x + \frac{1}{2}\int_{-1}^{1} (t-x)\left[\frac{3}{4}t + \frac{1}{4}\right] dt = x + \frac{1}{2}\left[\frac{(3)t^3}{4\cdot 3} + \frac{t^2}{8} - \frac{3xt^2}{8} - \frac{xt}{4}\right]_{-1}^{1}$

Always Check!

$= x + \frac{1}{2}\left[\left(\frac{1}{4} - \frac{-1}{4}\right) + \cancel{\left(\frac{1}{8} - \frac{1}{8}\right)}^{0} - \cancel{\left(\frac{3x}{8} - \frac{3x}{8}\right)}^{0} - \left(\frac{x}{4} - \frac{-x}{4}\right)\right] = x + \frac{1}{2}\left[\frac{1}{2} - \frac{x}{2}\right] = \frac{3x}{4} + \frac{1}{4}$ ✓