# Lecture #3: Branch Cuts, Analytic Continuation, and Residue Theorem

## I. Branch Cuts

### A. Multiple Branch Points

1. Consider $f(z) = (z^2-1)^{\frac{1}{2}}$

a. Find singularities (branch points)

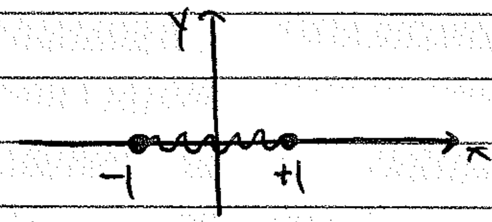
i. $f(z) = (z+1)^{\frac{1}{2}}(z-1)^{\frac{1}{2}}$ ii. $\frac{df(z)}{dz} = \frac{(z+1)^{\frac{1}{2}}}{2(z-1)^{\frac{1}{2}}} + \frac{(z-1)^{\frac{1}{2}}}{2(z+1)^{\frac{1}{2}}}$

iii. Thus, singularities at $z=+1$, $z=-1$.

iv. Check $z=\infty$: $f(\frac{1}{w}) = (\frac{1}{w^2}-1)^{\frac{1}{2}} = \frac{1}{w}(1-w^2)^{\frac{1}{2}}$

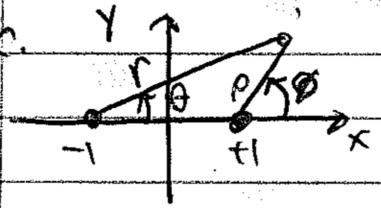
$\lim_{w\to 0} f(\frac{1}{w}) = \infty \rightarrow z=\infty$ is also a singularity.

b.




Let's try a branch cut from $z=-1$ to $z=+1$.

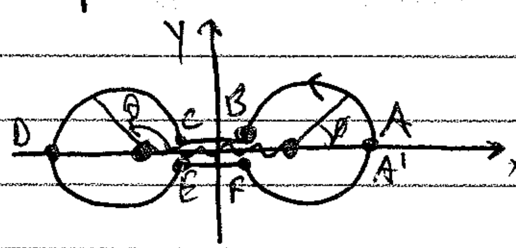
c.




i. Define: $z+1 = \rho e^{i\phi}$, $z-1 = r e^{i\theta}$

ii. Thus $f(z) = \rho^{\frac{1}{2}} r^{\frac{1}{2}} e^{i\left(\frac{\phi+\theta}{2}\right)}$

d.




| Point | $\theta$ | $\phi$ | $\left(\frac{\theta+\phi}{2}\right)$ |
|---|---|---|---|
| A | 0 | 0 | 0 |
| B | 0 | $\pi$ | $\pi/2$ |
| C | 0 | $\pi$ | $\frac{\pi}{2}$ |
| D | $\pi$ | $\pi$ | $\pi$ |
| E | $2\pi$ | $\pi$ | $\frac{3\pi}{2}$ |
| F | $2\pi$ | $\pi$ | $\frac{3\pi}{2}$ |
| A' | $2\pi$ | $2\pi$ | $2\pi$ |

$\leftarrow f(z) = +\sqrt{3}$ (at A)

$\leftarrow f(z) = +\sqrt{3}$ (at A')

e. NOTE:

i. Phase at B the same as F, C the same as E $\Rightarrow$ branch cut

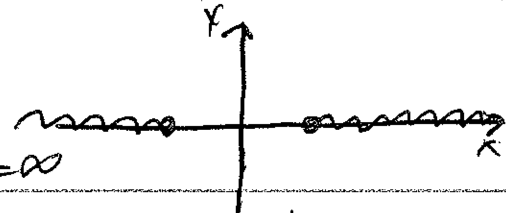
ii. Phase at A' is $2\pi$, so this branch cut has made $f(z)$ single-valued.

f. By passing around both poles (each contribution $\pi$ phase change), to phase change is $2\pi$. Cannot encircle just one pole!

Hawes ③

I. A. (Continued)

2. Alternative Branch Cuts




a. $z=-\infty$ to $z=-1$, $z=1$ to $z=\infty$

b. Also prevents encircling a single pole.

## II. Analytic Continuation:

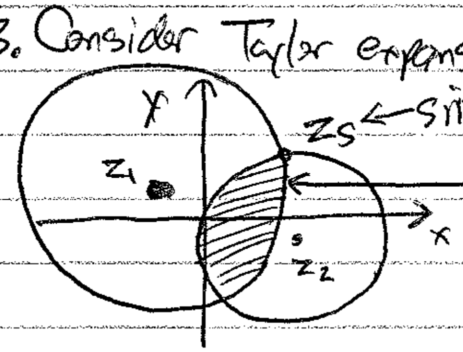
### A. Extending the Region of Analyticity

1. An analytic function $f(z)$ may only converge in a limited region.
   a. ie, a Taylor expansion converges within radius of convergence, excluding the nearest singularity
2. But, if we can find another analytic function $g(z)$, with a different region of convergence, and that function is the same in the region of overlapping convergence, we can extend analytic region.

### B. Showing two functions are the same

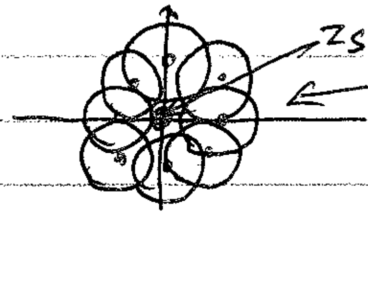
1. Properties: a. Taylor series coefficients are proportional to $f^{(n)}(z)$
   b. Analytic functions have all orders of derivatives, independent of direction.
   c. Values of $f(z)$ on a single finite line segment (with $z_0$ as an interior point) suffices to determine all derivatives $f^{(n)}(z_0)$.
2. Therefore: If two analytic functions coincide on a finite line segment, they are the same function.

3. Consider Taylor expansions about two points, $z_1$ & $z_2$




a. If we can show $f_1(z)$ and $f_2(z)$ coincide on a segment in region of overlap, we can extend analytic region!

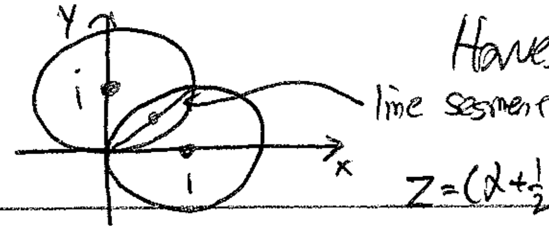
4. This process can be repeated




Can show analytic in entire angular region.

II. B. Continued

Hanes ③




$z = (\alpha + \tfrac{1}{2})(1+i)$

$y = mx + b$

$x = \alpha$

$m = (1+i) \quad b = \frac{1+i}{2}$

5. Ex: $f_1(z) = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$

$$f_2(z) = \sum_{n=0}^{\infty} i^{n-1} (z-i)^n$$

a. Substitute for $z$ and expand about $\alpha = 0$:

$$f_1 = \sum_{n=0}^{\infty} (-1)^n \left[(1+i)\alpha - \frac{1-i}{2}\right]^n$$

$$f_2 = \sum_{n=0}^{\infty} i^{n-1} \left[(1+i)\alpha + \frac{1-i}{2}\right]^n$$

Use binomial theorem to expand about $\alpha = 0$.

b. Show expansion coefficients are the same (as $\alpha$ varies along line).

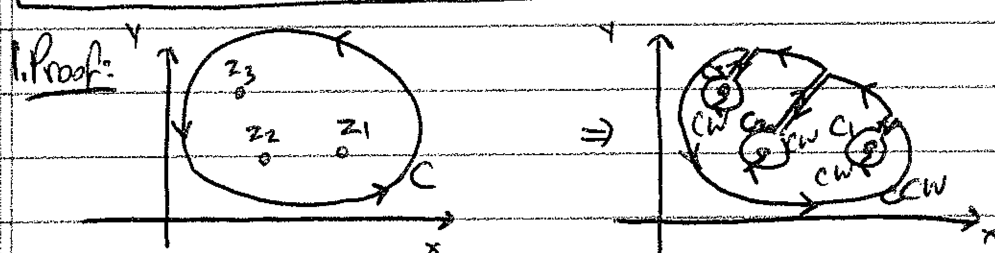
# III. Evaluating Contour Integrals using Residues

## A. Residue Theorem:

If $C$ is a positively oriented closed contour within and on which $f(z)$ is analytic except for a finite number of singular points $z_k$ $(k = 1, 2, \dots, n)$ interior to $C$, then

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} [f(z)]$$

1. Proof:




a. $\oint_C f(z)\,dz$ $\Rightarrow$ b. $\underset{(CCW)}{\oint_C} f(z)\,dz + \sum_{k=1}^{n} \underset{(CW)}{\oint_{C_k}} f(z)\,dz = 0$

c. Converting $C_k$ integrals to CCW changes sign

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz$$

d. Consider a Laurent expansion of $f(z)$ about $z = z_k$ (singularity)

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_k)^n$$

e. From previous result, (see Lect 1, III C. 4), $\oint_C (z-z_0)^n dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n=-1, \end{cases}$

So $$\oint_{C_k} f(z)dz = 2\pi i \, a_{-1_k}$$

where $a_{-1_k}$ is the residue about the singular point $z=z_k$.

f. Thus $$\oint_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} a_{-1_k} = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k}\left[f(z)\right].$$

B. Computing Residues.

1. Simple pole: If $f(z)$ can be written $f(z) = \frac{\phi(z)}{z-z_0}$, where $\phi(z)$ is analytic at $z=z_0$ and $\phi(z_0) \neq 0$, then

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0)}$$

2. Pole of order m: If $f(z) = \frac{\phi(z)}{(z-z_0)^m}$, then

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \frac{\frac{\partial^{(m-1)}}{\partial z^{(m-1)}} \phi(z)}{(m-1)!}}$$

3a. Simple pole:

If it is not simple to express $f(z) = \frac{\phi(z)}{(z-z_0)}$, you may take

$$\boxed{\operatorname*{Res}_{z=z_0}\left[f(z)\right] = \lim_{z\to z_0}\left[(z-z_0)f(z)\right]}$$ where you may need L'Hopital's Rule.

b. Pole of order m: $$\boxed{\operatorname*{Res}_{z=z_0}\left[f(z)\right] = \frac{1}{(m-1)!}\lim_{z\to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}\left\{(z-z_0)^n f(z)\right\}\right]}$$

III. B. (Continued)

4. Examples:

a. $f(z) = \frac{1}{4z+1}$ at $z = -\frac{1}{4}$: $f(z) = \frac{1}{4(z+\frac{1}{4})}$ $\Rightarrow$ $\left.\begin{matrix}\phi(z) = \frac{1}{4} \\ z_0 = -\frac{1}{4}\end{matrix}\right\}$ $\phi(z_0) = \boxed{\frac{1}{4}}$

b. $f(z) = \frac{1}{\sin z}$ at $z = 0$: $\lim_{z\to 0}\left(\frac{z}{\sin z}\right) = \lim_{z\to 0}\frac{1}{\cos z} = \boxed{1}$

L'Hopital's Rule: $\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$

c. $f(z) = \frac{\cot(\pi z)}{z(z+2)}$ at $z = 0$: i. Expand $\cot(\pi z) = \frac{1}{\pi z} - \frac{\pi z}{3} - \dots$

ii. Expand $\frac{1}{(z+2)} = \frac{1}{2(1+\frac{z}{2})} = \frac{1}{2}\left[1 - \frac{z}{2} + \frac{z^2}{4} - \dots\right]$

iii. Thus $f(z) = \frac{1}{z}\left[\frac{1}{\pi z} - \frac{\pi z}{3}\dots\right]\left(\frac{1}{2}\right)\left[1 - \frac{z}{2} + \frac{z^2}{4} - \dots\right]$

$= \frac{1}{2\pi z^2} - \frac{1}{4\pi z} + \mathcal{O}(1)$

iv. Coefficient of $\frac{1}{z}$ is $\boxed{-\frac{1}{4\pi}}$

d. $f(z) = e^{-\frac{1}{z}}$ at $z = 0$: (essential singularity)

i. $e^{-\frac{1}{z}} = 1 - \frac{1}{z} + \frac{1}{2z^2} - \frac{1}{3!z^3} + \dots$ ii. Coefficient of $\frac{1}{z}$ is $\boxed{-1}$

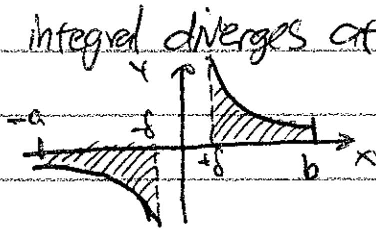
## C. Cauchy Principle Value

Def: For a real function $f(x)$ with isolated singularity at $x = x_0$, for example $f(x) = \frac{g(x)}{x-x_0}$,

$$P\int_{-\infty}^{\infty}\frac{g(x)}{x-x_0}dx \equiv \lim_{\delta\to 0}\left[\int_{-\infty}^{x_0-\delta}dx\,\frac{g(x)}{x-x_0} + \int_{x_0+\delta}^{\infty}dx\,\frac{g(x)}{x-x_0}\right]$$

1. Ex: $\int_{-a}^{b}\frac{dx}{x}$ where $a>0$ and $b>0$, integral diverges at $x=0$.

a. $P\int_{-a}^{b}\frac{dx}{x} = \lim_{\delta\to 0}\left[\int_{-a}^{-\delta}\frac{dx}{x} + \int_{\delta}^{b}\frac{dx}{x}\right]$



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III C1. (Continued)

b. $\int_{\delta}^{b} \frac{dx}{x} = \ln x \Big|_{\delta}^{b} = \ln b - \ln \delta$

c. $\int_{-a}^{-\delta} \frac{dx}{x} = \int_{a}^{\delta} \frac{dx}{x} = \ln x \Big|_{a}^{\delta} = \ln \delta - \ln a$

$y = -x$

d. Thus $P\int_{-a}^{b} \frac{dx}{x} = \cancel{\ln \delta} - \ln a + \ln b - \cancel{\ln \delta} = \ln b - \ln a$

e. In this case, symmetry leads to cancellation of $[-\delta, 0]$ and $[0, \delta]$.

2. Ex: $I = \int_{0}^{\infty} \frac{\sin x}{x} dx$ logarithmic divergence at $x=0$

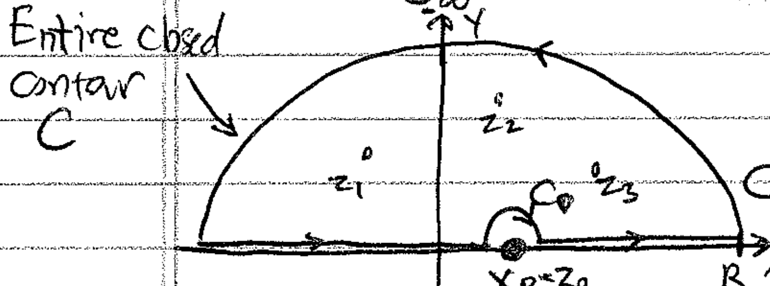
a. Let $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$, so $I = \int_{0}^{\infty} \frac{e^{ix} - e^{-ix}}{2ix} dx \overset{\downarrow}{=} \lim_{\delta \to 0} \int_{\delta}^{\infty} \frac{e^{ix} - e^{-ix}}{2ix} dx$

$= \lim_{\delta \to 0} \left[ \int_{\delta}^{\infty} \frac{e^{ix}}{2ix} dx - \int_{-\delta}^{-\infty} \frac{e^{+iy} dy}{2iy} \right] = \boxed{P\int_{-\infty}^{\infty} \frac{e^{ix}}{2ix} dx}$

$y = -x$

## D. Using Principal Value in Evaluation of Contour Integrals

1. Consider $P\int_{-\infty}^{\infty} f(x)\, dx$ of a function with an isolated simple pole at $x = x_0$.




a. Replace $f(x)$ with equivalent $f(z)$.

b. $\oint_C f(z)\,dz = P\int_{-\infty}^{\infty} f(z)\,dz + \int_{C_0 \,(\text{cw})} f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{\text{Res}} [f(z)]$

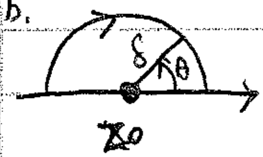
By Residue Theorem.

2. Compute: $\int_{C_0} f(z)\,dz$

Laurent Expansion

a. Since $f(x)$ has simple pole at $x_0$, $f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \ldots$

b.




$z - z_0 = \delta e^{i\theta}$
$dz = \delta i e^{i\theta} d\theta$

$\int_{C_0} f(z)\,dz = \int_{\pi}^{0} d\theta\, i \delta e^{i\theta} \left[ \frac{a_{-1}}{\delta e^{i\theta}} + a_0 + a_1 \delta e^{i\theta} + \ldots \right]$

III. D. 2. (Continued)

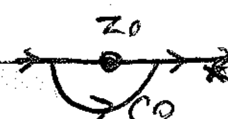
c. Comparing $\lim_{\delta\to 0}\int_{\pi}^{0} d\theta\, i\left[a_{-1}+a_0\delta e^{i\theta}+a_1(\delta e^{i\theta})^2+\ldots\right] = 0 - i\pi a_{-1} = -i\pi \underset{\underset{z=z_0}{\operatorname{Res}} f(z)}{\underset{\uparrow}{a_{-1}}}$

d. Thus $\int_{C_0} f(z)\,dz = -i\pi \underset{z=z_0}{\operatorname{Res}} f(z)$ ← negative sign since CW contour.

3. Typically, by taking $R\to\infty$, $\int_{C_R} f(z)\,dz \to 0$.

4. Thus, solving for Principle Value:

$$\boxed{P\int_{-\infty}^{\infty} f(z)\,dz = +i\pi \underset{z=z_0}{\operatorname{Res}} f(z) + 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{\operatorname{Res}} f(z)}$$

5. NOTE: a. If $C_0$ is taken below pole 

$\Rightarrow$ Sign of $\underset{z=z_0}{\operatorname{Res}} f(z)$ changes.

b. But, sum of singularities includes $z_0$!

6. Plemelj Relation (Not in text)

$$\boxed{\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dx \frac{f(x)}{x-(x_0\pm i\epsilon)} = P\int_{-\infty}^{\infty} dx \frac{f(x)}{x-x_0} \pm i\pi f(x_0)}$$

sign: CCW → +
CW → −

# E. Pole Expansion of Meromorphic Functions

1. Def: Meromorphic Function: Analytic function $f(z)$ with only isolated poles throughout finite complex plane.

2. **Mittag-Leffler's Theorem:** For $f(z)$ analytic at $z=0$ and all other finite points except isolated simple poles,

$$\boxed{f(z) = f(0) + \sum_{n=1}^{\infty} b_n\left(\frac{1}{z-z_n}+\frac{1}{z_n}\right)}$$

a. Expansion of $f(z)$ with each term from a different pole of $f(z)$.

III. E. (Continued)

3. Ex: Pole Expansion of $\tan z$

a. Take $\tan z = \dfrac{\sin z}{\cos z} = \dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$

b. Poles occur at $\cos z = 0 \rightarrow z_n = \pm \dfrac{(2n+1)\pi}{2}$

c. Find Residue at $z_n$:

$$\operatorname*{Res}_{z=z_n} f(z) = \lim_{z \to z_n} \left[ \frac{\left(z - \frac{(2n+1)\pi}{2}\right) \sin z}{\cos z} \right] \overset{\text{L'Hopital's Rule}}{=} \lim_{z \to z_n} \frac{\sin z + \cancel{\left(z - \frac{(2n+1)\pi}{2}\right)}^{0} \cos z}{-\sin z} \qquad z_n = \frac{(2n+1)\pi}{2}$$

$$= \frac{\sin z_n}{-\sin z_n} = \boxed{-1} \text{ for all } n!$$

d. Using Mittag-Leffler's Theorem,

$$\tan(z) = \cancel{\tan(0)}^{0} + \sum_{n=0}^{\infty} (-1) \left[ \frac{1}{z - \frac{(2n+1)\pi}{2}} + \frac{1}{\frac{(2n+1)\pi}{2}} \right] + \sum_{n=0}^{\infty} (-1) \left[ \frac{1}{z + \frac{(2n+1)\pi}{2}} - \frac{1}{\frac{(2n+1)\pi}{2}} \right]$$

$z_n = +\frac{(2n+1)\pi}{2}$ $\qquad$ $z_n = -\frac{(2n+1)\pi}{2}$

e. Can be rearranged to:

$$\boxed{\tan(z) = 2z \left( \frac{1}{\left(\frac{\pi}{2}\right)^2 - z^2} + \frac{1}{\left(\frac{3\pi}{2}\right)^2 - z^2} + \dots \right)}$$

F. Counting Poles and Zeros

1. To learn number of poles and zeros of otherwise analytic function $f(z)$, use $\dfrac{f'(z)}{f(z)}$

a. If $f(z) = (z - z_0)^{\mu} g(z)$ with $g(z_0) \neq 0$ ($\mu$th order pole or zero)

then $$\frac{f'(z)}{f(z)} = \frac{\mu (z - z_0)^{\mu - 1} g(z) + (z - z_0)^{\mu} g'(z)}{(z - z_0)^{\mu} g(z)} = \frac{\mu}{(z - z_0)} + \frac{g'(z)}{g(z)}$$

b. Thus $\dfrac{f'(z)}{f(z)}$ has a simple pole at $z = z_0$ with residue $\mu$.

2. Thus $\boxed{\oint_C \dfrac{f'(z)}{f(z)} dz = 2\pi i (N_f - P_f)}$

$N_f$ = number of zeros (multiplied by order)

$P_f$ = number of poles (multiplied by order).