Lecture #4: Evaluation of Definite Integrals by Contour Integration

I. Definite Integrals

A. Trigonometric Integrals over $(0, 2\pi)$

1. Consider integrals of the form $I = \int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$

   where f is finite and single valued for all values of θ.

2. Change Variables: a. $z = e^{i\theta}$, $dz = i e^{i\theta} d\theta$

   b. In the complex plane, integration over $0 \le \theta \le 2\pi$ is CCW along unit circle.

   c. Substitute: $\boxed{d\theta = -i \frac{dz}{z}}$, $\boxed{\sin\theta = \frac{z - z^{-1}}{2i}}$, $\boxed{\cos\theta = \frac{z + z^{-1}}{2}}$

3. Thus
$$I = -i \oint_C \frac{1}{z} f\left(\frac{z-z^{-1}}{2i}, \frac{z+z^{-1}}{2}\right) dz$$
where C is unit circle.

4. By Residue Thm,
$$\boxed{I = -i(2\pi i) \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} \left[\frac{f(z)}{z}\right]}$$
For all poles within unit circle.

5. Ex: $I = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}$, $|a| < 1$

   a. Convert from θ to z:
$$I = -i \oint_C \frac{dz}{z\left[1 + a\left(\frac{z+z^{-1}}{2}\right)\right]} = -i \frac{2}{a} \oint_C \frac{dz}{z^2 + \left(\frac{2}{a}\right)z + 1}$$

   b. Compute poles
$$z^2 + \left(\frac{2}{a}\right)z + 1 = 0 \Rightarrow \begin{cases} z_1 = -\left(\frac{1 + \sqrt{1-a^2}}{a}\right) \\ z_2 = -\left(\frac{1 - \sqrt{1-a^2}}{a}\right) \end{cases}$$

   c. NOTE: For $|a| < 1$, $|z_1| > 1$ and $|z_2| < 1$

   So, only pole z_2 is inside unit circle.

   d. Thus
$$I = -i \frac{2}{a} \oint \frac{dz}{(z-z_1)(z-z_2)} = -i \frac{2}{a} 2\pi i \operatorname*{Res}_{z=z_2} \left[\frac{\phi(z_2)}{(z - z_2)}\right]$$
where $\phi(z) = \frac{1}{z - z_1}$

I. A.5. (Continued)

e. $I = -i\frac{2}{a}2\pi i\left(\frac{1}{z_2 - z_1}\right) = \frac{4\pi}{a}\left[\frac{1}{-\left(\frac{1-\sqrt{1-a^2}}{a}\right)+\left(\frac{1+\sqrt{1-a^2}}{a}\right)}\right] = \frac{\overset{2}{\cancel{4}}\pi}{\cancel{a}}\cdot\frac{\cancel{a}}{\cancel{2}\sqrt{1-a^2}}$

$$\boxed{I = \frac{2\pi}{\sqrt{1-a^2}}, \quad |a|<1}$$

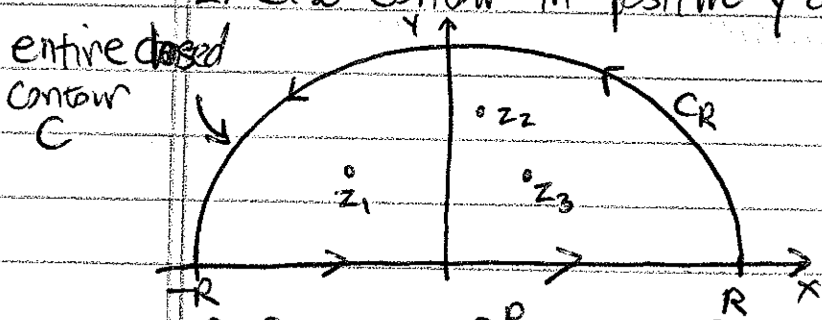
B. Integrals over $(-\infty, \infty)$

1. Consider $I = \int_{-\infty}^{\infty} f(x)\,dx$

a. $f(z)$ is analytic in upper half-plane except for isolated poles. (Assume no poles on real axis)

b. $\lim_{|z|\to\infty}[z f(z)] = 0$ for upper half-plane $(0 \le \theta \le \pi)$

2. Close contour in positive y-direction:



By Residue Theorem

a. $\oint_C f(z)dz = \int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{\text{Res}}[f(z)]$

3. Negligible contribution of arc C_R:

$$\boxed{\text{If } \lim_{R\to\infty} z f(z) = 0 \text{ for all } z = Re^{i\theta} \text{ over } \theta_1 \le \theta \le \theta_2, \text{ then } \lim_{R\to\infty}\int_{C_R} f(z)dz = 0}$$

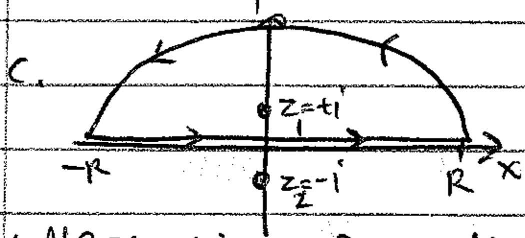
4. Thus $\boxed{\int_{-\infty}^{\infty} f(z)dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{\text{Res}}[f(z)]}$ upon limit $R \to \infty$.

I. B. (Continued)

5. Ex: $I = \int_0^\infty \frac{dx}{1+x^2}$

a. Since integrand is even, we can extend to $(-\infty, \infty)$ $\quad I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dz}{1+z^2}$

b. Thus $f(z) = \frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$ $\Rightarrow$ Poles at $z_1 = +i$, $z_2 = -i$

c. 

$\oint_C f(z)dz = \int_{-\infty}^{\infty} f(z)dz + \int_{C_R} f(z)dz$

d. NOTE: $\lim_{R\to\infty} z f(z) = \lim_{R\to\infty} Re^{i\theta} \frac{1}{1+R^2e^{i2\theta}} = 0$ for $0 \le \theta \le \pi$

Thus $\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$

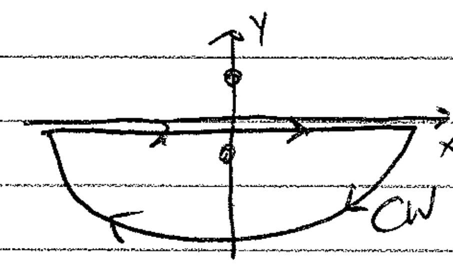
e. Thus, by Residue Theorem

$I = \left(\frac{1}{2}\right)\int_{-\infty}^{\infty} f(z)dz = \left(\frac{1}{2}\right) 2\pi i \operatorname*{Res}_{z=z_1} f(z) =$ ← only singularity interior to C.

f. $\operatorname*{Res}_{z=z_1} f(z) = \frac{1}{z_1+i} = \frac{1}{2i}$

g. Thus $I = \frac{1}{2} \cancel{2}\pi \cancel{i} \frac{1}{\cancel{2}\cancel{i}} = \boxed{\frac{\pi}{2}}$

h. NOTE: We could also have closed contour in lower half-plane, where CW contour leads to a negative sign in Residue Thm.



C. Integrals with Complex Exponentials

1. Consider $I = \int_{-\infty}^{\infty} f(x) e^{iax} dx$

a. $f(z)$ is analytic in upper half-plane with only isolated poles

b. $\lim_{|z|\to\infty} f(z) = 0$ for $0 \le \theta \le \pi$ ← (less restrictive than previous condition)

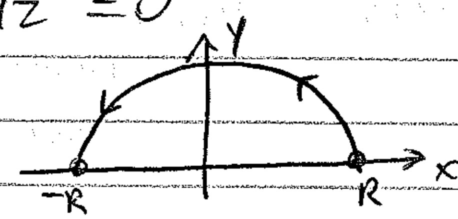
2. The second condition enables a contour closed in the upper half-plane (not the lower half-plane for $a>0$) to contribute negligibly, proven by Jordan's Lemma (Proof in text).

3. Jordan's Lemma

If $\lim_{R\to\infty} f(z) = 0$ for $z = Re^{i\theta}$ over $0 \le \theta \le \pi$, then

$$\lim_{R\to\infty} \int_C e^{iaz} f(z)\,dz = 0$$

where $a \geq 0$ and C is a semicircle in upper half-plane.



4. Thus $\oint_C f(z)e^{iaz}\,dz = \int_{-\infty}^{\infty} f(z)e^{iaz}\,dz + \cancel{\int_{C_R} f(z)e^{iaz}\,dz} = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} e^{iaz} f(z)$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} f(x)e^{iax}\,dx = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k}\left[e^{iaz} f(z)\right]}$$

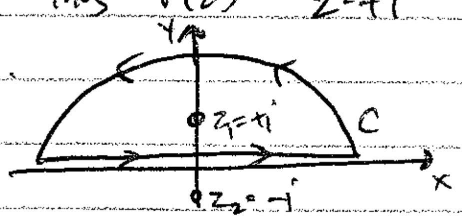
← Residues in upper half-plane

5. Ex: $I = \int_0^\infty \frac{\cos x}{x^2+1}\,dx$

a. By using $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, we may show $I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{1+x^2}$

b. Thus $f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$ → Poles at $z_1 = +i$, $z_2 = -i$.

c.



By result above,

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{iz}}{1+z^2}\,dz = \left(\frac{1}{2}\right)2\pi i \operatorname*{Res}_{z=z_1}\left[e^{iz}\frac{1}{z^2+1}\right]$$

d. $\operatorname*{Res}_{z=z_1}\left[\frac{e^{iz}}{(z+i)(z-i)}\right] = \frac{e^{i(i)}}{i+i} = \frac{e^{-1}}{2i}$ $\Rightarrow$ Thus $I = 2\pi i\left(\frac{1}{2}\right)\frac{1}{2ie} = \boxed{\frac{\pi}{2e}}$

I. (Continued) Hawes (5)

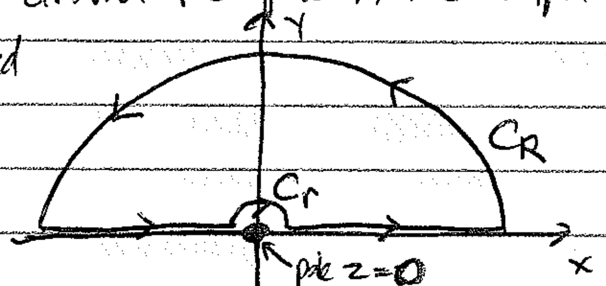
D. Pole on Contour of Integration

1. Consider $I = \int_0^{\infty} \frac{\sin x}{x} dx$

a. Using $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$, we can write $I = \int_{-\infty}^{\infty} \frac{e^{ix}}{2ix} dx$

But, this integral diverges at $x = 0$, so we need to integrate around the pole in the complex plane. ⇒ Use Principal Value

$$I = P \int_{-\infty}^{\infty} \frac{e^{iz}}{2iz} dz$$



b. $\oint_C f(z)dz = P\int_{-\infty}^{\infty} \frac{e^{iz}}{2iz} dz + \int_{C_r} \frac{e^{iz}}{2iz} dz + \cancel{\int_{C_R} \frac{e^{iz}}{2iz} dz} = 0$

$\int_{C_R} \to 0$ By Jordan's Lemma

$= 0$: No poles interior to C

c. $\int_{C_r} \frac{e^{iz}}{2iz} dz = \lim_{r \to 0} \int_{\pi}^{0} \frac{\cancel{e^{ire^{i\theta}}}^{1}}{2i\cancel{re^{i\theta}}} (\cancel{ire^{i\theta}} d\theta) = \int_{\pi}^{0} \frac{d\theta}{2} = -\frac{\pi}{2}$

d. Thus $\boxed{I = \frac{\pi}{2}}$

E. Handling Real Integrals over $(0, \infty)$ without Even Symmetry

1. To handle $I = \int_0^{\infty} \frac{dx}{1+x^2}$, we exploited even symmetry, $I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

2. For $I = \int_0^{\infty} \frac{dx}{1+x^3}$, we need to exploit other symmetries in θ.

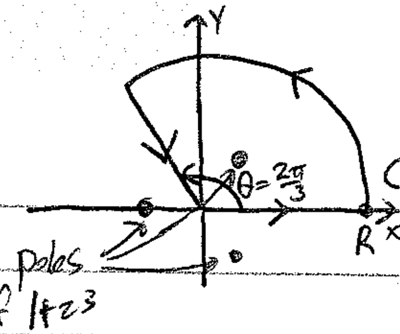
a. NOTE: Along $\theta = \frac{2\pi}{3}$, function is the same as along real axis.

$z = re^{i\theta}$

i) $\theta = 0$ $\quad z^3 = (re^{i0})^3 = r^3$

ii) $\theta = \frac{2\pi}{3}$ $\quad z^3 = (re^{i\frac{2\pi}{3}})^3 = r^3 e^{i2\pi} = r^3$

I. E. (Continued)



3. Thus, we may integrate around a circular sector

4. Result: $I = \frac{2\pi}{3\sqrt{3}}$ (see text for details)

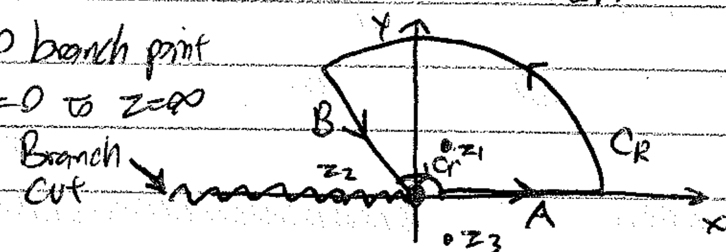
F. Avoiding Branch Points

1. Consider $I = \int_0^\infty \frac{\ln x \, dx}{1+x^3}$

2. Taking $f(z) = \frac{\ln z}{1+z^3}$, simple poles at $z_1 = e^{i\frac{\pi}{3}}$, $z_2 = e^{i\pi}$, $z_3 = e^{i\frac{5\pi}{3}}$

   and a branch point at $z=0$ due to $\ln z$ factor.

   a. Contour must avoid $z=0$ branch point and branch cut from $z=0$ to $z=\infty$



   b. By Residue Theorem — Closed contour $C = A + C_R + B + C_r$

$$\oint_C f(z)dz = \underbrace{\int_0^R f(z)dz}_{=I} + \underbrace{\int_{C_R} f(z)dz}_{①} + \underbrace{\int_B f(z)dz}_{②} + \underbrace{\int_{C_r} f(z)dz}_{③} = 2\pi i \operatorname*{Res}_{z=z_1}[f(z)]$$

① c. Since $\lim_{R\to\infty} z\left(\frac{\ln z}{1+z^3}\right) = 0$ over $0 \le \theta \le \frac{2\pi}{3}$, then $\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$

② d. Using $z = re^{i\frac{2\pi}{3}}$, $dz = dr\,e^{i\frac{2\pi}{3}}$, $\int_B \frac{\ln z\, dz}{1+z^3} = \int_\infty^0 \frac{(\ln r + i\frac{2\pi}{3})}{1+r^3} e^{i\frac{2\pi}{3}} dr = -e^{i\frac{2\pi}{3}} I + \frac{i2\pi}{3} e^{i\frac{2\pi}{3}} \int_\infty^0 \frac{dr}{1+r^3}$

   i. NOTE: Here we must specify branch $-\pi \le \theta \le \pi$! $\Rightarrow \theta = \frac{2\pi}{3}$ along B.

   ii. Integral $\int_\infty^0 \frac{dr}{1+r^3} = \frac{-2\pi}{3\sqrt{3}}$ (see I.E.4. above)

③ e. $\lim_{r\to 0} \int_{C_r} f(z)dz = \lim_{r\to 0} \int_{2\pi/3}^0 \frac{(\ln r + i\theta)}{1+r^3 e^{i3\theta}} ire^{i\theta} d\theta = 0$ since $\lim_{r\to 0} r\ln r = 0$.

f. $\operatorname*{Res}_{z=z_1}[f(z)] = \lim_{z\to z_1} \frac{(z-z_1)\ln z}{z^3+1} = \left.\frac{\ln z + \cancel{\frac{(z-z_1)}{z}}}{3z^2}\right|_{z=z_1} = \frac{i\frac{\pi}{3}}{3e^{2\pi i/3}}$ — L'Hopital's Rule

I. F. (Continued)

f. Thus, putting together the results

$$I + 0 + \left(-e^{i\frac{2\pi}{3}}\right)I + \frac{i2\pi}{3}e^{i\frac{2\pi}{3}}\left(\frac{-2\pi}{3\sqrt{3}}\right) = 2\pi i \frac{i\frac{\pi}{3}}{3e^{\frac{2\pi i}{3}}}$$

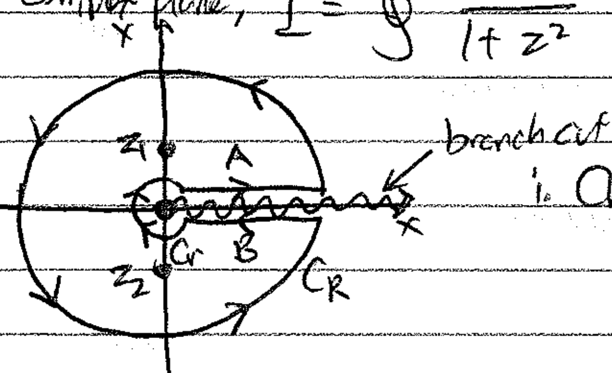
g. Some tedious algebra solves for $\boxed{I = -\frac{2\pi^2}{27}}$

G. Exploiting Branch Cuts:

1. Consider $I = \int_0^\infty \frac{x^p\,dx}{1+x^2}$ $\quad 0<p<1$

a. For complex plane, $I = \oint \frac{z^p\,dz}{1+z^2}$ i) where $z=0$ is a branch point. ii) $z_1 = +i$, $z_2 = -i$ are simple poles

Closed contour $C = A + C_R + B + C_r$



i. Choose branch $0 \le \theta \le 2\pi$

$$\oint_C f(z)dz = \underbrace{\int_\varepsilon^R f(z)dz}_{=I} + \underbrace{\int_{C_R} f(z)dz}_{①} + \underbrace{\int_B f(z)dz}_{②} + \underbrace{\int_{C_r} f(z)dz}_{③} = 2\pi i \sum_{K=1}^{2} \operatorname{Res}_{z=z_k}[f(z)]$$

b. NOTE: $\lim_{\varepsilon\to 0}\lim_{R\to\infty} \int_\varepsilon^R \frac{z^p\,dz}{1+z^2} = \lim_{\varepsilon\to 0}\lim_{R\to\infty}\int_\varepsilon^R \frac{r^p\,dr}{1+r^2} = I$

① c. $\lim_{R\to\infty}\int_{C_R} f(z)dz = \lim_{R\to\infty}\int_0^{2\pi} \frac{R^p e^{ip\theta}}{1+R^2e^{i2\theta}}\, iRe^{i\theta}d\theta = 0$ since $\lim_{R\to\infty}\frac{R^{p+1}}{R^2} = 0$.

② d. On segment B, $\theta = 2\pi$ on this branch, so $z^p = r^p e^{i2\pi p}$, and

$$\int_R^\varepsilon \frac{r^p e^{i2\pi p}}{1+r^2e^{i4\pi}}dr = -e^{i2\pi p}\int_\varepsilon^R \frac{r^p\,dr}{1+r^2} = -e^{i2\pi p} I$$

③ e. $\lim_{r\to 0} \int_{2\pi}^{0} \frac{r^p e^{ip\theta}}{1+r^2e^{i2\theta}}\, ire^{i\theta}d\theta = 0$

I. G.1 (Continued)

c. $\underset{z=z_1}{\text{Res}}\left[\frac{z^p}{(z+i)(z-i)}\right] = \frac{e^{i\frac{p\pi}{2}}}{2i}$, $\underset{z=z_2}{\text{Res}}\left[f(z)\right] = \frac{e^{i\frac{3p\pi}{2}}}{-2i}$

$z_1 = i = e^{i\frac{\pi}{2}}$ $\qquad z_2 = -i = e^{i\frac{3\pi}{2}}$

<u>NOTE:</u> On this branch $\theta = \frac{3\pi}{2}$, not $\theta = -\frac{\pi}{2}$!

2. Thus

a. $I + 0 + (-e^{i2\pi p})I + 0 = 2\pi i \frac{e^{i\frac{p\pi}{2}} - e^{i\frac{3p\pi}{2}}}{2i}$

b. Can be simplified to $\boxed{I = \frac{\pi}{2\cos\left(\frac{p\pi}{2}\right)}}$