Lecture #7: Dirichlet Series, Infinite Products, Asymptotic Series, & Method of Steepest Descent

I. Dirichlet Series

A. 1. Def: Dirichlet Series, $\boxed{S(s) = \sum_n \frac{a_n}{n^s}}$

a. Ex: Riemann Zeta Function, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

2. Evaluation of functions by a limit of a parameter.

a. Ex: Using contour integration with $\pi \cot(\pi z)$, you may obtain

$$S(a) = \sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi \coth(\pi a)}{2a} - \frac{1}{2a^2}$$

b. Taking $\lim_{a \to 0}$, we obtain result for $\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2)$.

i. $$\lim_{a\to 0}\left[\frac{\pi \coth(\pi a)}{2a} - \frac{1}{2a^2}\right] \underset{\uparrow}{=} \lim_{a \to 0}\left[\frac{\pi}{2a}\left(\cancel{\frac{1}{\pi a}} + \frac{\pi a}{3} - \frac{(\pi a)^3}{45} + \dots\right) - \cancel{\frac{1}{2a^2}}\right] = \frac{\pi^2}{6}$$

Laurent expansion

3. Other useful Dirichlet Series are

a. $\eta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} = (1-2^{1-s})\zeta(s)$

b. $\lambda(s) = \sum_{n=0}^{\infty} (2n-1)^{-s} = (1-2^{-s})\zeta(s)$

c. $\beta(s) = \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-s}$

4. Typically, such integrals can be expressed in terms of Bernoulli numbers, or evaluated by contour integration methods.

II. Infinite Products

A. Properties of Infinite-Product Representations

1. Def: Infinite Product $\boxed{P = \prod_{n=1}^{\infty}(1+a_n)}$

2. NOTE: Taking log converts to an infinite sum, $\ln ab = \ln a + \ln b$

$$\ln P = \ln \prod_{n=1}^{\infty}(1+a_n) = \sum_{n=1}^{\infty} \ln(1+a_n)$$

3. Convergence Thm:

If $0 \le a_n < 1$, the infinite products $\prod_{n=1}^{\infty}(1+a_n)$ and $\prod_{n=1}^{\infty}(1-a_n)$

i. converge if $\sum_{n=1}^{\infty} a_n$ converges

ii. diverge if $\sum_{n=1}^{\infty} a_n$ diverges

4. For convergence, note that a. $(1+a_n) \le e^{a_n} = 1 + a_n + \frac{a_n^2}{2!} + \frac{a_n^3}{3!} + \ldots$

   b. Thus partial product $p_n = (1+a_1)(1+a_2)(1+a_3)\cdots(1+a_n) \le e^{a_1}e^{a_2}e^{a_3}\cdots e^{a_n} = e^{S_n}$

   where $S_n = \sum_{i=1}^{n} a_i$ is the partial sum.

   c. Letting $n\to\infty$, $\prod_{n=1}^{\infty}(1+a_n) \le e^{\sum_{n=1}^{\infty} a_n}$ ← If sum converges, so does product.

5. Ex: Product Expansion of $\sin z$ converges

a. $\sin z = z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right)$

b. Check $\sum_{n=1}^{\infty} a_n = \frac{z^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{z^2}{\pi^2}\zeta(2) = \frac{z^2}{\cancel{\pi^2}}\frac{\cancel{\pi^2}}{6} = \boxed{\frac{z^2}{6}}$ convergent!

III. Asymptotic Series

A. Properties

1. Asymptotic series arise in many approximations in physics:
   a. Ex: WKB expansion
2. Can be used to compute numerically a variety of functions
3. Although asymptotic series formally diverge, they provide a basis for accurate estimation by partial sums.

4. Two types of integrals lead to asymptotic series:
   a. $I_1(x) = \int_x^\infty e^{-u} f(u)\,du$
   b. $I_2(x) = \int_0^\infty e^{-u} f\left(\frac{u}{x}\right)du$ expand f as Taylor series in $\frac{u}{x}$.

B. Example: The Exponential Integral

1. Consider the Exponential Integral, $\boxed{E_i(x) \equiv \int_{-\infty}^{x} \frac{e^u}{u}\,du}$
2. By taking $u \to -u$ and $x \to -x$, this can be rewritten,

$$\boxed{E_1(x) \equiv \int_x^\infty \frac{e^{-u}}{u}\,du}$$ ← We want to evaluate this integral for large values of x.

3. A convergent series expansion can be found (see chap 13),

$$E_1(x) = -\gamma - \ln x - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n\, n!}$$

but this is not useful for computing $E_1(x)$ when x is large!

⇒ We want an expression to evaluate $E_1(x)$ for large x!

4a. Generalize $I(x,p) = \int_x^\infty \frac{e^{-u}}{u^p}\,du$

b. Integrate by parts $I = \int_x^\infty \frac{e^{-u}}{u^p}\,du = \left[\frac{e^{-u}}{u^p}\right]_x^\infty - p\int_x^\infty \frac{e^{-u}\,du}{u^{p+1}}$

$u = \frac{1}{u^p},\quad dv = e^{-u}du$

$du = \frac{-p}{u^{p+1}},\quad v = -e^{-u}$

$$= \frac{e^{-x}}{x^p} - p\int_x^\infty \frac{e^{-u}\,du}{u^{p+1}}$$

c. Continuing with n integration by parts yields

$$\boxed{I(x,p) = e^{-x}\left[\frac{1}{x^p} - \frac{p}{x^{p+1}} + \frac{p(p+1)}{x^{p+2}} - \dots + (-1)^{n-1}\frac{(p+n-2)!}{(p-1)!\,x^{p+n-1}}\right] + (-1)^n\frac{(p+n-1)!}{(p-1)!}\int_x^\infty \frac{e^{-u}}{u^{p+n}}\,du}$$

5. Check Convergence using Ratio Test

$$\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty}\left[\frac{(p+n)!}{\cancel{(p-1)!}}\,\frac{1}{x^{p+n-1}}\,\frac{\cancel{(p-1)!}\,x^{p+n-2}}{(p+n-1)!}\right] = \lim_{n\to\infty}\frac{(p+n)!}{(p+n-1)!}\,\frac{1}{x}$$

$$= \lim_{n\to\infty}\frac{p+n}{x} = \infty \longrightarrow \boxed{\text{Divergent!}}$$

6. __BUT__, Compute remainder for a partial sum, $R_n \equiv I - S_n$

a. $$R_n(x,p) = I(x,p) - S_n(x,p) = (-1)^n\frac{(p+n)!}{(p-1)!}\int_x^\infty \frac{e^{-u}}{u^{p+n+1}}\,du$$

b. Investigate $|R_n(x,p)|$

i. Let $u = v + x$, so $\int_x^\infty \frac{e^{-u}}{u^{p+n-1}}\,du = e^{-x}\int_0^\infty \frac{e^{-v}\,dv}{(v+x)^{p+n-1}}$

$$= \frac{e^{-x}}{x^{p+n-1}}\int_0^\infty e^{-v}\underbrace{\left(1+\frac{v}{x}\right)^{-p-n+1}}\,dv$$

For large x, $1+\frac{v}{x} \simeq 1 \Rightarrow \int_0^\infty \Rightarrow 1.$

c. Thus $\boxed{|R_n(x,p)| = \frac{(p+n)!}{(p-1)!}\,\frac{e^{-x}}{x^{p+n-1}}}$ ← For a sufficiently large x, we can make this arbitrarily small!

7. Therefore, the asymptotic series, although divergent as an infinite sum, can be used to obtain an accurate estimate from a partial sum!

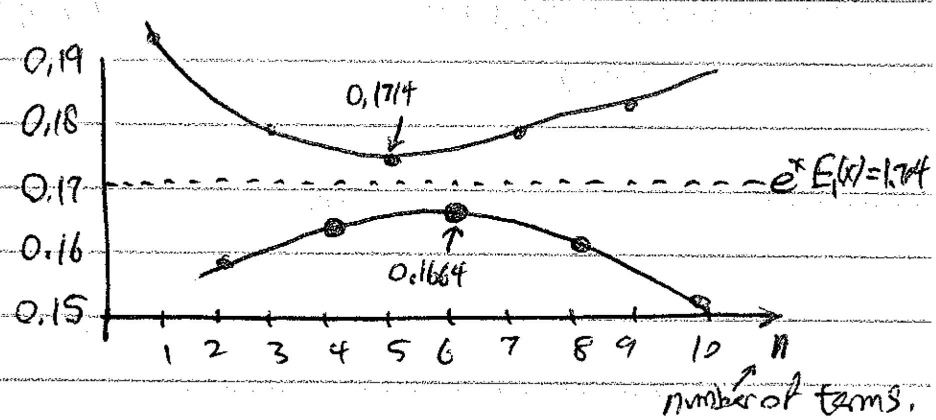
8. For $E_1(x)$, we obtain

a. $$\boxed{e^x E_1(x) = \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \dots + (-1)^n \frac{n!}{x^{n+1}}}$$ for n terms.

b. For $x=5$, we obtain

$$\boxed{0.1664 \le e^x E_1(x)\Big|_{x=5} \le 0.1741}$$

1.704



C. Asymptotic Series Definition

1. Consider
   a. function $f(x)$
   b. partial sum $S_n(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots + \frac{a_n}{x^n}$
   c. remainder $R_n(x)$

2. Power Series representation: (Ex)

$$\boxed{x^n R_n(x) = x^n \left[f(x) - S_n(x) \right]}$$

3. Asymptotic Expansion of $f(x)$ has properties:
   a. $\lim_{x\to\infty} x^n R_n(x) = 0$ for fixed n
   b. $\lim_{n\to\infty} x^n R_n(x) = \infty$ for fixed x

4. For power series, $R_n(x) \approx x^{-n-1}$ $\Rightarrow f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n} = S_n(x)$

$\Rightarrow$ Equal only in limits $x \to \infty$ and n finite.

III. C. (Continued)

5. Properties:

a. Asymptotic expansions may be multiplied together

b. " " " " integrated term by term

c. But, term by term differentiation is valid only in special circumstances

IV. Method of Steepest Descent

A. Basic Concept:

1. Determine asymptotic behavior of $f(t)$ in limit of large (real) t.

2. Necessary Conditions:

a. $f(t) = \int_C F(z,t)\,dz$ (Complex contour integral) & $F(z,t)$ analytic

b. C can be deformed such that, for large t, dominant contribution to integral is a range of z near z_0, where $|F(z_0,t)|$ is maximum on the path.

c. Steepest Descent: Path through z_0 follows most rapid decrease in $|F|$.

d. For large t, contribution near z_0 asymptotically approaches $f(t)$.

3. These conditions occur for many physics applications (Gamma, Bessel Fncs).

B. Saddle Points

1. Recall, for analytic function $F(z,t)$, neither real nor imaginary part can have an extremum within region of analyticity.

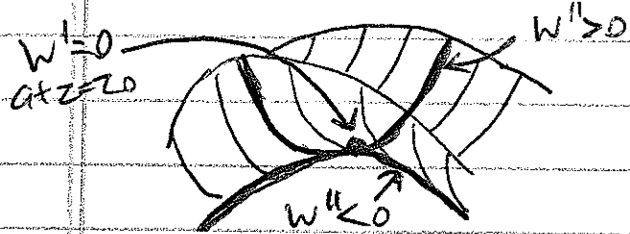
a. Jensen's Thm: $|F(z,t)|$ also has no extremum in analytic region.

2. Take $F(z,t) = e^{w(z,t)} = e^{u(z,t) + i v(z,t)}$ where

a. $w(z,t)$ is an analytic function

b. $F(z,t)$ is nonzero within region of interest.

3. Saddle Points: Although u (and v) cannot have an extremum, it can have a saddle point with $w'=0$ in all directions at z_0, but $w''(z_0)\geq 0$ in some directions, $w''(z_0)<0$ in other directions.



b. Expand $W(z,t)$ about saddle point at $z=z_0$:

$$W(z,t)=W(z_0,t)+\cancel{W'(z_0,t)}^{\,0}(z-z_0)+W''(z_0,t)\frac{(z-z_0)^2}{2}+\ldots$$

c. Since $W''(z_0,t)$ is just a complex constant, we may write

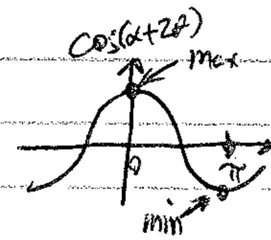
$$W''(z_0,t)=|W_0''|e^{i\alpha} \quad \text{in polar form}$$

d. Also, writing $z-z_0=re^{i\theta}$, we obtain

$$W(z,t)=W_0+\tfrac{1}{2}|W_0''|e^{i\alpha}r^2e^{i2\theta}=W_0+\tfrac{1}{2}|W_0''|r^2e^{i(\alpha+2\theta)}$$

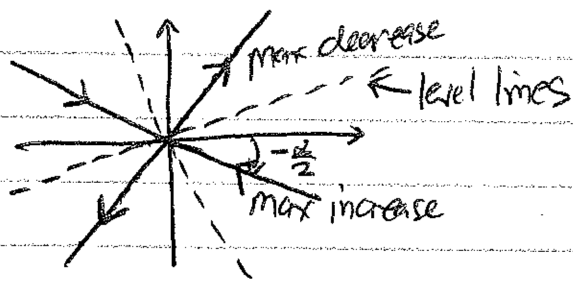
$$=W_0+\tfrac{1}{2}|W_0''|r^2\left[\cos(\alpha+2\theta)+i\sin(\alpha+2\theta)\right]$$

4. Direction of Steepest descent:



a. For the real part, max increase at $\alpha+2\theta=2n\pi$, or $\theta=\frac{-\alpha}{2}$ and $\theta=-\frac{\alpha}{2}+\pi$

b. Maximum decrease at $\alpha+2\theta=(2n+1)\pi$, or $\theta=\frac{-\alpha}{2}+\frac{\pi}{2}$ or $\theta=\frac{-\alpha}{2}+\frac{3\pi}{2}$



c. Level lines: u is constant (to 2nd order) along $\theta=-\frac{\alpha}{2}+\left(\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}\right)$

c. Maximum increase and decrease of v (imaginary) are along level lines of u (and vice versa).

IV. B. (Continued)

5. Optimum Contour a. Contour C passes through z_0 along path of steepest descent in U
   b. This path is along constant V, so e^{iV} will not produce oscillatory behavior.

C. Method of Steepest Descent

1. Assume contributions to integral are dominated by range $0 < r \leq a$ along both directions of optimum path.
2. Choose θ along optimum path, so $dz = e^{i\theta} dr$
3. $f(t) = \int_C F(z,t)\,dz \approx 2\int_0^a e^{W_0 + \frac{1}{2}|W_0''| r^2 [\cancelto{-1}{\cos(\alpha+2\theta)} + i\cancelto{0}{\sin(\alpha+2\theta)}]} e^{i\theta} dr$

   both directions away from z_0

$$\boxed{f(t) \approx 2e^{W_0 + i\theta} \int_0^a e^{-\frac{1}{2}|W_0''| r^2} dr}$$

4. If $|W_0''|$ is sufficiently large (when t is large), the exponential decrease in integrand is rapid enough to take $a \to \infty$ with negligible change.
5. Recall $e^{W_0} = e^{W(z_0,t)} = F(z_0,t)$ and use

   $\int_0^\infty e^{-\frac{|W_0''|}{2} r^2} dr = \sqrt{\frac{\pi}{2|W_0''|}}$ to obtain

$$\boxed{f(t) \approx F(z_0,t)\, e^{i\theta} \sqrt{\frac{2\pi}{|W_0''|}}}$$

where $\theta = -\frac{\arg(W_0'')}{2} + \left(\frac{\pi}{2} \text{ or } \frac{3\pi}{2}\right)$

6. Need to check, a posteriori, that only the region near z_0 yields a significant contribution to the integral.

IV. D. Example: Asymptotic Form of Gamma Function

1. Let us estimate $\Gamma(t+1) = t!$ for real t large.

$$\Gamma(t+1) = \int_0^\infty \rho^t e^{-\rho} d\rho$$

a. $\rho^t e^{-\rho} = e^{\ln \rho^t} e^{-\rho} = e^{t \ln \rho - \rho}$

b. Substitute $\rho = zt$ to convert to contour integral $F(z,t)$

$$\Rightarrow \Gamma(t+1) = t^{t+1} \int_0^\infty e^{t(\ln z - z)} dz$$

2. Thus, $w(z,t) = t(\ln z - z)$

a. $w' = \frac{dw}{dz} = \frac{t}{z} - t = 0$ at $z_0 = 1$

b. $w'' = \frac{-t}{z^2}$ At $z_0 = 1$, $w_0'' = -t = |t| e^{i\pi} \Rightarrow \alpha = \pi$

c. Direction of steepest descent: $\theta = -\frac{\alpha}{2} + \left(\frac{\pi}{2} \text{ or } \frac{3\pi}{2}\right) = 0$ or π

i. Choose $\theta = 0$ (along real axis, consistent with original integral)

3. For $F(z,t) = e^{t(\ln z - z)}$, we obtain $F(z_0,t) = e^{-t}$

$$\Gamma(t+1) \approx e^{-t} e^{i(0)} \sqrt{\frac{2\pi}{t}}\, t^{t+1} = \boxed{\sqrt{2\pi}\, t^{t+\frac{1}{2}} e^{-t} \approx \Gamma(t+1)}$$

Leading term of Stirling's expansion of Gamma Function.