Lecture #8 Dispersion Relations and Bessel Functions

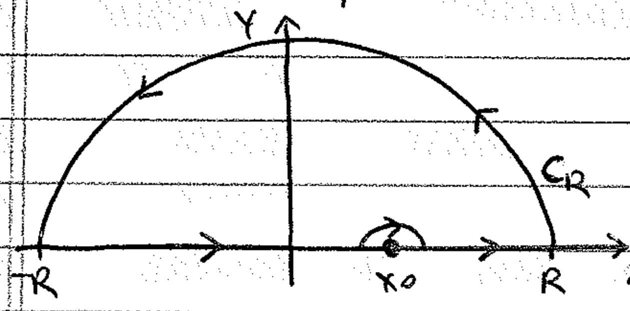
I. Dispersion Relations

A. Basic Concept

1. Generalization: Dispersion Relations: A pair of equations giving the real part of a function as an integral of its imaginary part, and vice versa.
   a. Integral analog of Cauchy-Riemann differential equations.

B. Derivation of Dispersion Relations

1. Consider a complex function $f(z)$ analytic for $y \geq 0$.



a. By Residue Theorem

$$\oint_C \frac{f(z)}{z-x_0}\,dz = 0 \leftarrow \text{no poles inside } C.$$

b. $$\oint_C \frac{f(z)}{z-x_0}\,dz = \underbrace{P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}\,dx - i\pi f(x_0)}_{\text{By Plemelj Relation (see Lec\#3, III.D.6.)}} + \cancelto{0}{\int_{C_R}\frac{f(z)}{z-x_0}\,dz} = 0$$

c. Require that $\lim_{|z|\to\infty} z\left(\frac{f(z)}{z-x_0}\right) = \lim_{|z|\to\infty} f(z) = 0$ such that $\lim_{R\to\infty}\int_{C_R}\frac{f(z)}{z-x_0}\,dz = 0$.

2. Thus
$$\boxed{f(x_0) = \frac{1}{i\pi}P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}\,dx}$$
NOTE: $f(x)$ is a complex function of a real variable x.

3. Separate real and imaginary part of $f(x) = U(x) + iV(x)$

$$U(x_0) + iV(x_0) = \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{V(x)}{x-x_0}\,dx - i\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{U(x)}{x-x_0}\,dx$$

I.B. (Continued)

4. Dispersion Relations: $\boxed{U(x_0) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{V(x)}{x-x_0}dx, \quad V(x_0) = -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{U(x)}{x-x_0}dx}$

5. Integral Transforms:

a. $U(x)$ and $V(x)$ are related by an integral transform

b. The particular transform here is the Hilbert transform.

i) The Hilbert transform is its own inverse (with minus sign)

C. Symmetry Properties:

1. Reality Condition: $\boxed{f(-x) = f^*(x)}$

a. Occurs when $f(x)$ arises as a Fourier transform of a real function

2. Using $f(x) = U(x) + iV(x)$, we obtain $U(-x) + iV(-x) = U(x) - iV(-x)$, so

$U(-x) = U(x)$ U is even in x

$V(-x) = -V(x)$ V is odd in x

3. This condition can be used to convert $\int_{-\infty}^{\infty}$ to $\int_0^{\infty}$, yielding

$$\boxed{U(x_0) = \frac{2}{\pi} P\int_0^{\infty} \frac{x V(x)}{x^2 - x_0^2}dx \qquad V(x_0) = -\frac{2}{\pi} P\int_0^{\infty} \frac{x_0 U(x)}{x^2 - x_0^2}dx}$$

a. ⇒ Original Kronig-Kramers optical dispersion relations were in this form.

D. Kronig-Kramers Relations for Optical Dispersion

1. A function $f(x,t) = e^{i(kx-\omega t)}$ can describe an electromagnetic wave, with phase velocity $v = \frac{\omega}{k}$, wavenumber k, and frequency ω.

a. Index of refraction $\boxed{n \equiv \frac{ck}{\omega}}$

2. Maxwell's Equations yield $k^2 = \varepsilon \frac{\omega^2}{c^2}\left(1 + i\frac{4\pi\sigma}{\omega\varepsilon}\right)$ σ = conductivity, ε = electric permittivity

I. D. 2. (Continued)

a. NOTE: For $\sigma \neq 0$, k^2 has an imaginary part $\Rightarrow$ damping!

3. Consider the weak conductivity limit, $\frac{4\pi\sigma}{\omega\varepsilon} \ll 1$,

a. $$k = \varepsilon^{\frac{1}{2}} \frac{\omega}{c}\left(1 + i\frac{4\pi\sigma}{\omega\varepsilon}\right)^{\frac{1}{2}} \underset{\uparrow}{\simeq} \varepsilon^{\frac{1}{2}}\frac{\omega}{c} + i\frac{2\pi\sigma}{c\varepsilon^{\frac{1}{2}}}$$

binomial expansion

b. Thus $$f(x,t) = e^{i(kx-\omega t)} = e^{i\omega\left(\frac{x\varepsilon^{\frac{1}{2}}}{c} - t\right)} \underbrace{e^{-2\pi\sigma\frac{x\varepsilon^{\frac{1}{2}}}{c}}}_{\text{attenuation in } x!}$$

4. Kronig-Kramers Relations

a. Multiply by $\frac{c^2}{\omega^2}$, $$\boxed{n^2 = \frac{c^2k^2}{\omega^2} = \varepsilon + i\frac{4\pi\sigma}{\omega}}$$ Here σ, ε depend on ω.

b. To put into dispersion relation form, $f(\omega) = n^2(\omega) - 1$

~~so that $\lim_{|\omega|\to\infty} f(\omega) = 0$ for complex ω.~~

c. Thus $$\boxed{\mathrm{Re}\left[n^2(\omega_0) - 1\right] = \frac{2}{\pi} P\int_0^\infty \frac{\omega\, \mathrm{Im}\left[n^2(\omega) - 1\right]}{\omega^2 - \omega_0^2} d\omega}$$ and complementary equation for Im.

d. Knowledge of the absorption coefficient (imaginary part) at all frequencies specifies the real part of the index of refraction!

E. Parseval Relation

1. Parseval Relation: For $u(x)$ & $v(x)$ as Hilbert transforms, each square integrable ($\int_{-\infty}^{\infty} |u(x)|^2 dx$ is finite),

$$\boxed{\int_{-\infty}^{\infty} |u(x)|^2 dx = \int_{-\infty}^{\infty} |v(x)|^2 dx}$$

2. For Fourier Transforms, Parseval relation means energy is the same expressed in x or k!

II. Bessel Functions of the First Kind, $J_\nu(x)$

A. Bessel ODE

1. $$\boxed{x^2 J_\nu'' + x J_\nu' + (x^2 - \nu^2) J_\nu = 0}$$

a. Bessel Functions of the first kind are regular at $x=0$.

b. Second (linearly independent) solution $Y_\nu(x)$ are irregular at $x=0$.
$\Rightarrow$ Neumann functions

B. Generating Function for Integral Order

Laurent series

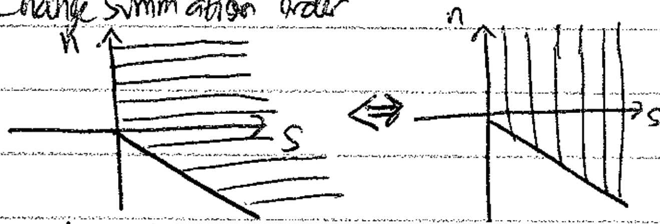
1. Generating Function $$\boxed{g(x,t) = e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n}$$

2. To obtain Frobenius (power series) solutions for $J_n(x)$, expand $e^{\frac{xt}{2}}$ and $e^{\frac{-x}{2t}}$ as Taylor series in exponent:

a. $$g(x,t) = e^{\frac{xt}{2}} e^{\frac{-x}{2t}} = \left[\sum_{r=0}^{\infty} \left(\frac{x}{2}\right)^r \frac{t^r}{r!}\right]\left[\sum_{s=0} (-1)^s \left(\frac{x}{2}\right)^s \frac{t^{-s}}{s!}\right] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{r+s} \frac{t^{r-s}}{r!\,s!}$$

b. Change index r to $n = r-s$ $$= \sum_{n=-s}^{\infty}\left[\sum_{s=0}^{\infty} \frac{(-1)^s}{(n+s)!\,s!}\left(\frac{x}{2}\right)^{n+2s}\right] t^n$$

c. Change summation order



$$\sum_{n=-\infty}^{\infty}\ \sum_{s=\max(0,-s)}^{\infty} \Leftrightarrow \sum_{n=-s}^{\infty}\ \sum_{s=0}^{\infty}$$

Lower limit $s = \max(0,-n)$

d. Thus, we obtain for $n \geq 0$ $$\boxed{J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!}\left(\frac{x}{2}\right)^{n+2s}}$$

Same as power series solution using Frobenius' Method.

II. B. (Continued)

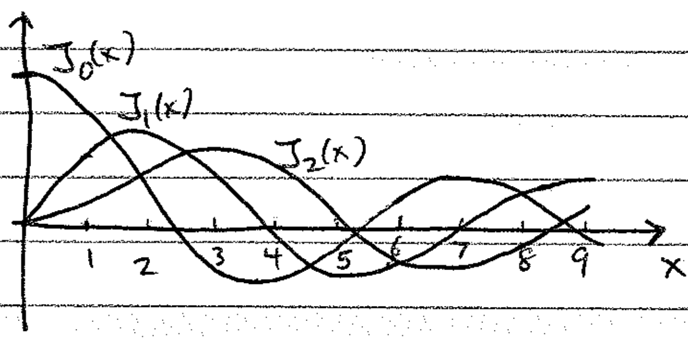
3. For $n<0$, $\boxed{J_{-n}(x) = (-1)^n J_n(x)}$ (integral n)
← linearly dependent!

4. For non-integer ν:

a. $$\boxed{J_\nu(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(\nu+s+1)} \left(\frac{x}{2}\right)^{\nu+2s}}$$ (for $\nu \neq -1, -2, \dots$)

b. Here $J_{-\nu}(x)$ and $J_\nu(x)$ are linearly independent!

5. J



C. Recurrence Relations

1. By taking $\frac{\partial}{\partial t} g(x,t)$ and $\frac{\partial}{\partial x} g(x,t)$, we may obtain:

a. Recurrence relation: $\boxed{J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)}$

b. Derivative relation: $\boxed{J_{n-1}(x) - J_{n+1}(x) = 2 J_n'(x)}$

2. Thus, all $J_n(x)$ can be computed from $J_0(x)$ and $J_1(x)$

3. By symmetry, $\boxed{J_0'(x) = -J_1(x)}$

D. Integral Representation of $J_n(x)$

1. By contour integration of $\frac{g(x,t)}{t^{m+1}}$ around $t=0$ (see lec#6, I.C.3),

$$\oint_C \frac{e^{\frac{x}{2}(t-\frac{1}{t})}}{t^{m+1}}\,dt = \oint_C \sum_n J_n(x)\, t^{n-m-1} = 2\pi i\, J_m(x) \qquad (m=n)$$

↖ contour around $t=0$

II. D. (Continued)

2. Taking the contour in complex t-plane to be the unit circle,
   a. $t = e^{i\theta} \quad dt = i\,d\theta\, e^{i\theta}$
   b. $e^{\frac{x}{2}(t-\frac{1}{t})} = e^{\frac{x}{2}(e^{i\theta}-e^{-i\theta})} = e^{ix\sin\theta}$
   c. Thus $2\pi i\, J_n(x) = \int_0^{2\pi} \frac{e^{ix\sin\theta}}{e^{i(n+1)\theta}}\, i e^{i\theta}\, d\theta = i\int_0^{2\pi} e^{i(x\sin\theta - n\theta)}\, d\theta$

3. For real x, $J_n(x) = \frac{1}{2\pi}\int_0^{2\pi}\left[\cos(x\sin\theta - n\theta) + i\sin(x\sin\theta - n\theta)\right]d\theta$

4. <u>Integral Representation:</u> From real part $\rightarrow$ $\boxed{J_n(x) = \frac{1}{2\pi}\int_0^{2\pi}\cos(x\sin\theta - n\theta)\,d\theta}$

5. Also, from imaginary part $\int_0^{2\pi}\sin(x\sin\theta - n\theta)\,d\theta = 0$

6. <u>Case for $J_0(x)$:</u> $\boxed{J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta}\,d\theta = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\,d\theta}$

E. <u>Solving Boundary Value Problems: Zeros of $J_n(x)$</u>

1. In cylindrical boundary value problems, the Bessel Functions solutions typically must yield $J_n(\rho) = 0$ at $\rho = a$.
   a. Thus, we must find zero values ρ_i where $J_n(\rho_i) = 0$!

2. No closed formulas for zeros of Bessel Functions
   a. Look up zeros in a table
   b. Compute zeros numerically.

3. But, there are asymptotic expansions for large orders, $\nu \gg 1$
   a. Ex: $\underset{\substack{\uparrow \\ \text{1st zero}}}{x_{\nu,1}} \sim \nu + 1.855757\,\nu^{\frac{1}{3}} + 1.033150\,\nu^{-\frac{1}{3}} - 0.00397\,\nu^{-1} - \ldots$

[See Abramowitz & Stegun, (9.5.14)]