

# Lecture #11 Jacobians and Differential Forms

Hanes 1

## I. Jacobians

1. Jacobians tell us how multidimensional integrals transform under a change of coordinate systems:  $dx_1 dx_2 dx_3 \Rightarrow \underbrace{r^2 \sin \theta}_{\text{Jacobian}} dr d\theta d\phi$

### A. General Form

1. Consider changing from  $(x_1, x_2, x_3)$  to  $(u_1, u_2, u_3)$

↙ Jacobian

a. Replace differential  $dx_1 dx_2 dx_3$  with  $J du_1 du_2 du_3$

b.  $dV = J du_1 du_2 du_3$  is a "volume" of length  $du_1, du_2, du_3$  in  $x_1, x_2, x_3$  space.

2. Compute displacement (in  $x_i$ -space) corresponding to change in  $u_j$ .

$$a. \underline{ds}(u_1) = \left[ \left( \frac{\partial x_1}{\partial u_1} \right) \hat{e}_1 + \left( \frac{\partial x_2}{\partial u_1} \right) \hat{e}_2 + \left( \frac{\partial x_3}{\partial u_1} \right) \hat{e}_3 \right] du_1$$

↙ in  $(x_1, x_2, x_3)$  space

$$ds(u_2) = \left[ \left( \frac{\partial x_1}{\partial u_2} \right) \hat{e}_1 + \left( \frac{\partial x_2}{\partial u_2} \right) \hat{e}_2 + \left( \frac{\partial x_3}{\partial u_2} \right) \hat{e}_3 \right] du_2, \text{ etc.}$$

b. NOTE: Partial derivatives  $\left( \frac{\partial x_i}{\partial u_j} \right)_{u_k}$  evaluated with  $u_k = \text{const}$  ( $k \neq j$ ).

3. In 2D, "Area" would be  $|ds(u_1)|$  times perpendicular component of  $ds(u_2)$ .

4. Matrix Form

$$\begin{pmatrix} \frac{ds(u_1)}{du_1} \\ \frac{ds(u_2)}{du_2} \\ \frac{ds(u_3)}{du_3} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \frac{\partial x_3}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_3}{\partial u_2} \\ \frac{\partial x_1}{\partial u_3} & \frac{\partial x_2}{\partial u_3} & \frac{\partial x_3}{\partial u_3} \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{pmatrix}$$

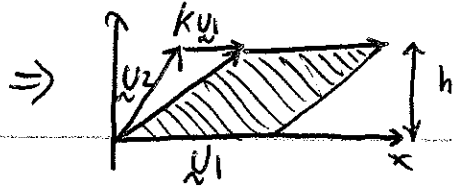
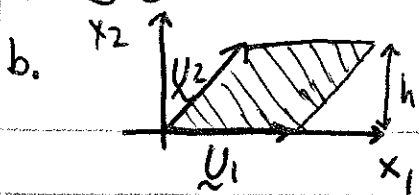
5. Jacobian can be computed by making the matrix above diagonal by adding multiples of one row to other rows

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Rightarrow \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix}$$

a. NOTE: This process does not change "volume" (or value of determinant.)

## I.A.5. (Continued)

Hines 2



"Area" =  $u_1 h$  is unchanged!

- c. "Volume" of Jacobian is just product of diagonal elements.  
 d. Since this process does not change the value of the determinant, it is the determinant that is the Jacobian.

6.

$$d\tau = J du_1 du_2 du_3, \quad J = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \frac{\partial x_1}{\partial u_3} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_2}{\partial u_3} \\ \frac{\partial x_3}{\partial u_1} & \frac{\partial x_3}{\partial u_2} & \frac{\partial x_3}{\partial u_3} \end{vmatrix} \equiv \frac{\partial(x_1, x_2, x_3)}{\partial(u_1, u_2, u_3)}$$

## 7. Ex: 2D Jacobian

a.  $(x, y) \Rightarrow (u, v) \quad dx dy = J du dv$

where  $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial y}{\partial v}\right) - \left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right)$

## B. Inverse of Jacobian

1. 3D  $dx_1 dx_2 dx_3 = J du_1 du_2 du_3$ , then  $du_1 du_2 du_3 = (J^{-1}) dx_1 dx_2 dx_3$ .

2. We can use  $u_1 = u_1(x_1, x_2, x_3)$ ,  $u_2 = u_2(x_1, x_2, x_3)$ , ...

to compute  $J^{-1} = \frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)}$

3. It is sometimes much easier to compute  $J^{-1}$  than  $J$ .

## 4. Ex: Spherical Coordinates:

a. Consider transformation from  $(r, \theta, \phi)$  to  $(x, y, z) \Rightarrow J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$

where  $r = \sqrt{x^2 + y^2 + z^2}$      $\theta = \cos^{-1}\left(\frac{z}{(x^2 + y^2 + z^2)^{1/2}}\right)$      $\phi = \tan^{-1}\left(\frac{y}{x}\right)$

b. But  $J^{-1} = \frac{\partial(r, \theta, \phi)}{\partial(x, y, z)}$  is the inverse where

$x = r \sin \theta \cos \phi$      $y = r \sin \theta \sin \phi$      $z = r \cos \theta$

## I. B. 4. (Continued)

HWes ③

$$C.S., j^{-1} = \begin{vmatrix} \sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\ r\cos\theta \cos\phi & r\cos\theta \sin\phi & -r\sin\theta \\ -r\sin\theta \sin\phi & r\sin\theta \cos\phi & 0 \end{vmatrix}$$

$$= r^2 \sin^3\theta \sin^2\phi + r^2 \cos^2\theta \sin\theta \cos^2\phi + r^2 \cos^2\theta \sin\theta \sin^2\phi + r^2 \sin^3\theta \cos^2\phi$$

$$= r^2 \sin\theta [\sin^2\theta + \cos^2\theta] = r^2 \sin\theta$$

d. Thus  $\boxed{dx dy dz = r^2 \sin\theta dr d\theta d\phi} \Rightarrow \boxed{dr d\theta d\phi = \frac{1}{r^2 \sin\theta} dx dy dz}$

## II. Differential Forms

### A. Introduction

1. Differential forms are a powerful generalization of vector and tensor analysis in curvilinear coordinate systems.
2. Enables treatment of topology (connectivity of spaces) in physics.

### B. Basic Quantities: Differentials $dx, dy, dz$

- a. Associated with linearly independent directions in space (orthogonal)

### 4. 1-forms $\omega = A(x, y, z) dx + B(x, y, z) dy + C(x, y, z) dz$

- a. Use as integrand in line integrals

### 5. 2-forms: $\omega = F(x, y, z) dx \wedge dy + G(x, y, z) dx \wedge dz + H(x, y, z) dy \wedge dz$

- a. Use as integrand of surface integrals

### 6. 3-forms: $\omega = K(x, y, z) dx \wedge dy \wedge dz$

- a. Use as integrand of volume integrals

### 7. Exterior Algebra (Grassman Algebra)

- a. "Wedge"  $\wedge$

- b. Permutational Antisymmetry

## II. A7. Continued

### c. Properties

$$(a\omega_1 + b\omega_2) \wedge \omega_3 = a\omega_1 \wedge \omega_3 + b\omega_2 \wedge \omega_3$$

$$(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$$

$$a(\omega_1 \wedge \omega_2) = (a\omega_1) \wedge \omega_2$$

$$dx_i \wedge dx_j = -dx_j \wedge dx_i$$

Hawes (4)

d. In general, any differential form can be reduced to a function multiplying a differential  $dx_i$  or wedge product  $dx_i \wedge dx_j$ .

$$(a dx_1) \wedge (b dx_2) = ab(dx_1 \wedge dx_2) = -ab(dx_2 \wedge dx_1)$$

e. By antisymmetry,  $dx_i \wedge dx_i = 0$

g. p-forms: a. A form of order  $p$  (contains  $p$  differentials)

b. Ordinary function or scalar is a 0-form.

c. Order  $p$  of form cannot be larger than dimension of space  $p \leq d$ .

9. Ex 1  $\omega = (3dx + 4dy - dz) \wedge (dx - dy + 2dz)$

$$= -3dx \wedge dy + 6dx \wedge dz + 4dy \wedge dx + 8dy \wedge dz - dz \wedge dx + dz \wedge dy$$

$$= -7dx \wedge dy + 7dy \wedge dz - 7dz \wedge dx = 7(-dx \wedge dy + dy \wedge dz - dz \wedge dx)$$

## B. Complementary Differential Forms

1. Complementary (or dual) form includes differentials not included in original form.

a. In dimension  $d$ , dual to  $p$ -form is a  $(d-p)$ -form.

2. Def: Hodge Star Operator  $*$

a. For  $*\omega$ , where  $\omega$  is a  $p$ -form, first form wedge product of  $(d-p)$  differentials not in  $\omega$ .

b. Sign of  $*\omega$  is determined by permutations needed to bring (indices of  $\omega$ ) followed by (indices of  $\omega^c$ ) to standard order.

## II. B. 2. (Continued)

Hawes (5)

- c. Formally, to find the complementary form requires the specification of the metric and selection of a reference order.
- d.  $*\omega$  also includes  $(-1)^\mu$ , where  $\mu$  is number of differentials in  $\omega$  whose metric tensor diagonal elements is  $-1$ .

### 3. Ex: Complementary Forms in $\mathbb{R}^3$

$$*1 = dx_1 \wedge dx_2 \wedge dx_3$$

$$*dx_1 = dx_2 \wedge dx_3, *dx_2 = dx_3 \wedge dx_1, \text{ etc.}$$

$$*(dx_1 \wedge dx_2) = dx_3, *(dx_3 \wedge dx_1) = dx_2, \text{ etc.}$$

$$*(dx_1 \wedge dx_2 \wedge dx_3) = 1$$

### 4. Ex: Minkowski Space $(dt, dx_1, dx_2, dx_3)$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

a.  $*1 = dt \wedge dx_1 \wedge dx_2 \wedge dx_3$

NOTE: 1 has no differentials, so  $\mu=0 \Rightarrow (-1)^\mu = 1$ .

b.  $*(dx_1 \wedge dx_2 \wedge dx_3) = -1$

NOTE: Here  $\mu=3$ , so  $(-1)^\mu = -1!$

c.  $*(dt \wedge dx_1) = -dx_2 \wedge dx_3$

NOTE:  $\mu=1$ , so  $(-1)^\mu = -1$

### 5. Connection to Vector Analysis

a.  $A = A_x dx + A_y dy + A_z dz, B = B_x dx + B_y dy + B_z dz$

b.  $*(A \wedge B) = (A_y B_z - A_z B_y) dx + (A_z B_x - A_x B_z) dy + (A_x B_y - A_y B_x) dz$

$$\Rightarrow *(A \wedge B) \iff \underline{A} \times \underline{B}$$

c.  $*(A \wedge B \wedge C) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \underline{A} \cdot (\underline{B} \times \underline{C})$

d. Generalizable to arbitrary dimension and metric.

### III. (Continued)

Pages 6

#### C. Differentiating Forms

##### 1. Def: Exterior Derivative $d\omega$

a. For  $\omega$  a  $p$ -form,  $\omega'$  a  $p'$ -form, and  $f$  a function (0-form)

$$d(\omega + \omega') = d\omega + d\omega' \quad (p=p')$$

$$d(f\omega) = (df) \wedge \omega + f d\omega$$

$$d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^p \omega \wedge d\omega'$$

$$d(d\omega) = 0$$

$$df = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j$$

b. NOTE: The derivative of a  $p$ -form is a  $(p+1)$ -form.

##### 2. Connection to Vector differential operators

a. d(0-form)  $\Rightarrow$  Gradient:  $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = (\nabla f)_x dx + (\nabla f)_y dy + (\nabla f)_z dz$

b. d(1-form)  $\Rightarrow$  Curl:

$$d(A_x dx + A_y dy + A_z dz) = (\nabla \times \underline{A})_x dy \wedge dz + (\nabla \times \underline{A})_y dz \wedge dx + (\nabla \times \underline{A})_z dx \wedge dy$$

c. d(2-form)  $\Rightarrow$  Divergence:

$$*d(B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy) = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \nabla \cdot \underline{B}$$

$$d. d(df) = 0 \Rightarrow \nabla \times (\nabla f) = 0 \quad (\text{Use } \underline{A} = \nabla f \text{ in (b) above})$$

$$e. d[d(A_x dx + A_y dy + A_z dz)] = \nabla \cdot (\nabla \times \underline{A}) dx \wedge dy \wedge dz = 0$$

(Use  $\underline{B} = \nabla \times \underline{A}$  in (c) above)

##### 3. Ex: Maxwell's Equations in Minkowski Space ( $dt, dx, dy, dz$ )

$$a. F = -E_x dt \wedge dx - E_y dt \wedge dy - E_z dt \wedge dz + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy \quad \text{replaces } \underline{E} \text{ \& } \underline{B}$$

### III. C3 (Continued)

Hawes 7

b. Source:  $J = \rho dx \wedge dy \wedge dz - J_x dt \wedge dy \wedge dz - J_y dt \wedge dx \wedge dz - J_z dt \wedge dx \wedge dy$   
 replaces  $\rho$  &  $J$ .

c. Use units with  $\epsilon_0 = \mu_0 = c = 1$ .

d. Thus  $\boxed{dF=0} \Rightarrow \nabla \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = 0$  and  $\nabla \cdot \underline{B} = 0$

e.  $\boxed{d(*F)=0} \Rightarrow \nabla \cdot \underline{E} = \rho$  and  $\nabla \times \underline{B} - \frac{\partial \underline{E}}{\partial t} = \underline{J}$

f.  $\boxed{dJ=0} \Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \underline{J} = 0$

### D. Integrating Forms

i. Integral of a 1-form  $\omega$  in 2D space over curve  $C$

$$\int_C \omega = \int_C [A_x dx + A_y dy]$$

a. For a parametric form of  $C$  as  $x(t)$  &  $y(t)$  from  $t_p$  to  $t_q$

$$\int_C \omega = \int_{t_p}^{t_q} [A_x(t) \frac{dx}{dt} + A_y(t) \frac{dy}{dt}] dt$$

b. For a conservative force,  $\omega$  is exact, such that  $\boxed{\omega = df(x,y)}$

c. Then  $\omega = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ , so  $\int_p^q \omega = f(Q) - f(P)$  (independent of path)

### 2. Surface Integrals of 2-forms

a.  $\int_S \omega = \int_S B(x,y) dx \wedge dy$  Here  $\int dx \wedge dy \Rightarrow \int dx dy$

b. Change of variables and Jacobian: i.  $x = au + bv \Rightarrow dx = a du + b dv$   
 $y = eu + fv \Rightarrow dy = e du + f dv$

ii.  $dx \wedge dy = (a du + b dv) \wedge (e du + f dv) = (af - be) du \wedge dv$

iii. But,  $af - be = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & b \\ e & f \end{vmatrix} = J \Rightarrow \boxed{dx \wedge dy = J du \wedge dv}$   
 naturally included!

## II. DZ (Continued)

c. NOTE: Since  $dx \wedge dy = -dy \wedge dx$ , the "area" in differential forms is oriented. Sign is necessary in Jacobian transformation. Hawes (8)

### 3. Stokes' Theorem:

- a. If  $R$  is a simply-connected region of a  $p$ -dimensional differentiable manifold in  $n$ -dimensional space ( $n \geq p$ )  
and b. If  $R$  has boundary  $\partial R$ , of dimension  $(p-1)$   
and c.  $\omega$  is a  $(p-1)$ -form defined on  $R$  and its boundary, <sup>with</sup> derivative  $d\omega$

$$\boxed{\int_R d\omega = \int_{\partial R} \omega}$$

- d. Applies for manifolds of any dimension (includes Stokes' & Gauss' Theorems from vector analysis).

### 4. Ex: Usual 3D case of Stokes' Theorem

- a.  $\omega = A_x dx + A_y dy + A_z dz$   
b.  $d\omega = \underbrace{\left[ \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right]}_{(\nabla \times \mathbf{A})_x} dy \wedge dz + \underbrace{\left[ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right]}_{(\nabla \times \mathbf{A})_y} dz \wedge dx + \underbrace{\left[ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right]}_{(\nabla \times \mathbf{A})_z} dx \wedge dy$   
c. Reference Order  $(dx, dy, dz) \Rightarrow dy \wedge dz \Rightarrow \underline{d\sigma}_x$ ,  $dz \wedge dx \Rightarrow \underline{d\sigma}_y$

$$\int_C \omega = \int_C (A_x dx + A_y dy + A_z dz) = \int_C \mathbf{A} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{\sigma}$$

Usual Vector Analysis version  
of Stokes' Theorem