Lecture #18: <u>Homogeneous & Inhomogeneous ODEs, Sturm-Liouville Theory</u>

I. <u>Second Solution for Linear Homogeneous Second-Order ODE (Continued)</u>

A. <u>Example</u>: Bessel's Equation for $n=0$.

1. $x^2 y'' + xy' + x^2 y = 0$

2. Putting it into standard form, $y'' + \underbrace{\frac{1}{x}}_{=P(x)} y' + \underset{Q(x)=1}{y} = 0$

3. From the Frobenius' Method Solution, indicial equation is $a_0(s^2 - n^2) = 0$

   a. So, $s^2 = 0 \Rightarrow \boxed{s=0}$ (from Lec#17, II.C.3.)

   b. Resulting series solution is $y_1(x) = J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - O(x^6)$

4. Now, use integral form for 2nd solution (Lec#17, III.C.2.),

   a. $\boxed{y_2(x) = y_1(x) \int^x \frac{\exp\left[-\int^{x_2} P(x_1)\,dx_1\right]}{\left[y_1(x_2)\right]^2}\,dx_2}$ $\quad$ <u>Wronskian Double Integral</u>

   b. $= J_0(x) \int^x \frac{\exp\left[-\int^{x_2}\frac{dx_1}{x_1}\right]}{\left[1 - \frac{x_2^2}{4} + \frac{x_2^4}{64} - \cdots\right]^2}\,dx_2$

   c. Numerator: $\exp\left[-\int^{x_2}\frac{dx_1}{x_1}\right] = \exp\left[-\ln x_2 + C_0\right] = \frac{e^{C_0}}{x_2} = \frac{C_1}{x_2}$

   d. Denominator: Use binomial expansion $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

   $\left[1 - \frac{x_2^2}{4} + \frac{x_2^4}{64} - \cdots\right]^{-2} = 1 - 2\left(\frac{-x_2^2}{4} + \frac{x_2^4}{64} - \right) + 3\left(\frac{-x_2^2}{4} + \frac{x_2^4}{64} - \cdots\right)^2 - \cdots$

   $\delta = \frac{-x_2^2}{4} + \frac{x_2^4}{64} - \cdots$

   $= 1 + \frac{x_2^2}{2} + \frac{5}{32}x_2^4 + \cdots$

   e. Thus

   $y_2(x) = J_0(x) \int^x \frac{C_1}{x_2}\left[1 + \frac{x_2^2}{2} + \frac{5}{32}x_2^4 + \cdots\right]dx_2 = \boxed{C_1 J_0(x)\left\{\ln x + \frac{x^2}{4} + \frac{5x^4}{128} + \cdots\right\}}$

5. Standard form of Second Solution of Bessel's Equation: <u>Neumann Function</u>

   $Y_0(x) = \frac{2}{\pi}\left[\ln x - \ln 2 + \gamma\right]J_0(x) + \frac{2}{\pi}\left\{\frac{x^2}{4} - \frac{3x^4}{128} + \cdots\right\}$

I. A. (Continued)

6. We can construct the Neumann function from a linear combination of our solutions $y_1(x)$ & $y_2(x)$

   a. $C_1 = \frac{2}{\pi}$, add $\frac{2}{\pi}\left[-\ln 2 + \gamma\right] J_0(x)$

   b. Multiply series expansion of $J_0(x)$ in Right-hand term.

7. <u>Note</u>: a. $Y_0(x)$ diverges at the origin $\Rightarrow$ <u>irregular solution</u>

      b. $J_0(x)$ converges at the origin $\Rightarrow$ <u>regular solution</u>

B. Summary of Solutions to Linear, Homogeneous, Second-Order ODEs

  1. Two Solutions $y_1(x)$ and $y_2(x)$ form complete solution

$$\boxed{y(x) = C_1 y_1(x) + C_2 y_2(x)}$$

  2. Point of expansion must be ordinary or regular singular point

  3. By Fuch's Theorem, one solution can always be obtained by Frobenius!

  4. <u>Wronskian double integral</u> yields 2nd, <u>linearly independent</u> solution.

  5. <u>No third linearly independent solution exists.</u>

II. Inhomogeneous Linear ODEs

1. General Form: $\boxed{y'' + P(x)y' + Q(x)y = F(x)}$

2. We assume homogeneous eq., $\boxed{F(x)=0}$ has been solved for $y_1(x)$ & $y_2(x)$

A. Variation of Parameters Method

1. Write the particular solution as $\boxed{y_p(x) = U_1(x)\, y_1(x) + U_2(x)\, y_2(x)}$

                                     Functions of x

2. $y_p' = U_1 y_1' + U_2 y_2' + (y_1 U_1' + y_2 U_2')$

3. We choose $U_1(x)$ & $U_2(x)$ such that $\boxed{y_1 U_1' + y_2 U_2' = 0}$

4. Thus

$$y_p'' = U_1 y_1'' + U_2 y_2'' + U_1' y_1' + U_2' y_2'$$

II.A. (Continued)

5. Substituting into ODE

a. $[U_1 y_1'' + U_2 y_2'' + U_1' y_1' + U_2' y_2'] + P(x)[U_1 y_1' + U_2 y_2'] + Q(x)[U_1 y_1 + U_2 y_2] = F(x)$

b. $U_1 \underbrace{[y_1'' + P(x)y_1' + Q(x)y_1]}_{= 0 \text{ (Homogeneous Solution)}} + U_2 \underbrace{[y_2'' + P(x)y_2' + Q(x)y_2]}_{= 0} + U_1' y_1' + U_2' y_2' = F(x)$

6. a. Thus, we are left with two equations to solve for U_1' & U_2'

$$y_1 U_1' + y_2 U_2' = 0$$
$$y_1' U_1' + y_2' U_2' = F(x)$$

b. NOTE:
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0 \quad \text{since } y_1 \text{ & } y_2 \text{ are linearly independent solutions}$$

c. Thus, there is a <u>unique, nontrivial solution</u> for U_1' & U_2'.

d. Simply integrate to obtain $U_1(x)$ & $U_2(x)$ and thus <u>particular solution</u> $y_p(x)$.

6. Ex: <u>Inhomogeneous</u> ODE $(1-x)y'' + x y' - y = (1-x)^2$

a. Solutions to homogeneous ODE are $y_1 = x$, $y_2 = e^x$.

b. Thus
$$x U_1' + e^x U_2' = 0$$
$$U_1' + e^x U_2' = F(x) = (1-x) \quad \leftarrow \text{Standard form!}$$

c. Solution yields $U_1' = 1 \quad \Rightarrow \quad U_1 = x$
$$U_2' = -x e^{-x} \quad \Rightarrow \quad U_2 = (x+1)e^{-x}$$

d. Form particular solution $y_p(x) = x[x] + (x+1)e^{-x}[e^x] = x^2 + x + 1$
(Can eliminate x because $y_1 = x$.)

e. General Solution is homogeneous solutions plus particular solution

$$\boxed{y(x) = C_1 x + C_2 e^x + x^2 + 1}$$

III. <u>Nonlinear Differential Equations (NDEs)</u>

1. Some physics involves <u>nonlinear differential equations</u>

a. Ex: Navier-Stokes equations for hydrodynamics.

III. (Continued)

2. Nonlinearity is responsible for development of _turbulence_

3. NDEs also lead to _chaos_, where a system is so _sensitive_ to
   initial conditions as to be _unpredictable_

4. Analytically, some NDEs can be solved by _clever_
   tricks that convert them to linear ODE's
   (Ex: Bernoulli & Riccati Eq's in text).

5. In general, behavior of NDEs is studied _numerically_.

IV. Sturm-Liouville Theory

A. Introduction

1. Many physics problems with ODEs have two properties:
   a. _Boundary Conditions must be satisfied_, ⟹ Define Hilbert space
   b. _Parameter must be chosen to satisfy BCs_ ⟹ Eigenvalue.

2. Ex: Standing Waves on a String Clamped at Ends

   a. $\frac{d^2\psi}{dx^2} + k^2\psi = 0$ where $\psi(x)$ is amplitude and k is parameter.

   b. Hilbert Space: Differentiable functions with $\psi = 0$ at ends.

   c. Eigenvalue Equation: $\mathcal{L}\psi = k^2\psi$ where $\mathcal{L} = -\frac{d^2}{dx^2}$

   d. We can solve this with a general _vector space_ approach
      i. Define a basis for Hilbert space
      ii. Define scalar product
      iii. Expand $\mathcal{L}$ and ψ in terms of basis
      iv. Solve resulting matrix equation

   e. But, since we can solve _the ODE_, we can do this more efficiently!

3. Ex: Standing Waves: ODE Solution with BCs.

   a. $\frac{d^2\psi}{dx^2} + k^2\psi = 0$ with $\psi(0) = \psi(\ell) = 0$.

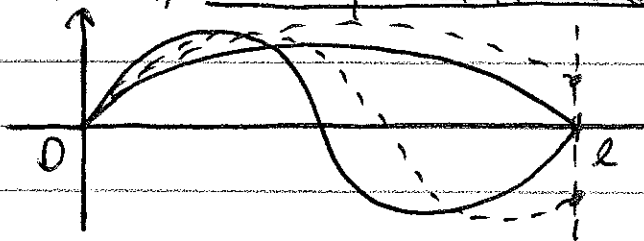b. General Solution: $\boxed{\psi(x) = A \sin kx + B \cos kx}$

c. Apply BCs: i. $\psi(0) = 0 = A(0) + B(1) \Rightarrow \boxed{B = 0}$

    ii. $\psi(\ell) = 0 = A \sin(k\ell) \Rightarrow k\ell = n\pi$ for any integer n.

d. Thus $\boxed{\psi_n(x) = A \sin\left(\dfrac{n\pi x}{\ell}\right)}$ where eigenvalue $\boxed{k^2 = \dfrac{n^2 \pi^2}{\ell^2}}$

4. <u>Discrete Solutions</u>: a. General Solution valid for any A, B, k.

  b. But, <u>boundary conditions restrict solutions to discrete k values.</u>

  c. Amplitude of Solution remains arbitrary $\Rightarrow$ homogeneous ODE.

5. <u>Normalizing Solutions</u>:

  a. Define Scalar Product $\langle f | g \rangle = \int_0^\ell f^*(x) g(x)\, dx$

  b. Require $\langle \psi_n | \psi_n \rangle = \underline{1}$.

  c. Thus, we obtain $\boxed{\psi_n(x) = \sqrt{\dfrac{2}{\ell}} \sin\left(\dfrac{n\pi x}{\ell}\right)}$

6. <u>Properties of Solution</u>:   Hermitian operator $\mathcal{L}$

  a. Hermitian property depends on $\mathcal{L}$ and definition of $\langle f | g \rangle$.

  b. Since this operator is Hermitian, it implies
      <u>real eigenvalues</u> and a <u>complete basis of orthogonal eigenfunctions</u>

B. <u>Hermitian Operators</u>

1. <u>Sturm-Liouville Theory</u> $\mathcal{L}\psi(x) = \lambda \psi(x)$

    where $\mathcal{L}(x) = p_0(x)\dfrac{d^2}{dx^2} + p_1(x)\dfrac{d}{dx} + p_2(x)$

a. We want to identify conditions under which $\mathcal{L}$ is <u>Hermitian</u>

IV B. (Continued)

2. **Self-Adjoint ODEs**

a. $\mathcal{L}$ is <u>self-adjoint</u> if $\boxed{p_0'(x) = p_1(x)}$

b. Thus $\mathcal{L}(x) = p_0 \dfrac{d^2}{dx^2} + p_0' \dfrac{d}{dx} + p_2 = \dfrac{d}{dx}\left[p_0 \dfrac{d}{dx}\right] + p_2$

c. Define $\langle f | g \rangle = \displaystyle\int_a^b f^*(x) g(x)\, dx$, so

$$\langle f | \mathcal{L}g \rangle = \int_a^b \left[f^*(p_0 g')' + f^* p_2 g\right] dx = \left[f^* p_0 g'\right]_a^b + \int_a^b \left(-f^{*'} p_0 g' + f^* p_2 g\right) dx$$

$$
\begin{array}{ll}
u = f^* & dv = (p_0 g')' \\
du = f^{*'} & v = p_0 g'
\end{array}
\qquad
\begin{array}{ll}
u = p_0 f^{*'} & dv = g' \\
du = (p_0 f^{*'})' & v = g
\end{array}
$$

$$= \left[f^* p_0 g' - f^{*'} p_0 g\right]_a^b + \int_a^b \underbrace{\left[(p_0 f^{*'})' + p_2 f^*\right]}_{= \mathcal{L} f^*} g \, dx = [\cdots]_a^b + \underbrace{\langle \mathcal{L}f | g \rangle}_{= \langle f | \mathcal{L}^\dagger g \rangle}$$

d. If boundary terms vanish, then $\mathcal{L}$ is self-adjoint, $\mathcal{L} = \mathcal{L}^\dagger$.

e. $\underset{\text{Classes of}}{\text{Boundary Conditions!}}$ i. <u>Dirichlet BC's</u>: $f(a) = g(a) = 0$, $f(b) = g(b) = 0$.

ii. <u>Neumann BC's</u>: $f'(a) = g'(a) = 0$, $f'(b) = g'(b) = 0$.

iii. <u>Periodic B.C.'s</u>: $f(a) = f(b)$, $g(a) = g(b)$.

f. <u>Orthogonality</u>: i. If f and g are eigenfunctions, with λ_f and λ_g, then

ii. $\langle f | \mathcal{L}g \rangle - \langle \mathcal{L}f | g \rangle = (\lambda_g - \lambda_f)\langle f | g \rangle = \left[f^* p_0 g' - f^{*'} p_0 g\right]_a^b$

iii. Then, if BC's vanish, $\langle f | g \rangle = 0$ for $\lambda_g \neq \lambda_f \Rightarrow$ <u>Orthogonality</u>!

C. **Making an ODE Self-Adjoint**

1. If an operator is not self-adjoint $\left[p_0'(x) = p_1(x)\right]$, it can be multiplied by a factor to make it <u>self-adjoint</u>.

a. $W(x)\mathcal{L}(x)\psi(x) = W(x)\lambda\psi(x)$ where $\boxed{W(x) = p_0^{-1} \exp\left[\int \dfrac{p_1(x)}{p_0(x)} dx\right]}$

Same eigenfunctions and eigenvalues.

IV. C. (Continued)

2. Then $\quad w(x)\mathcal{L}(x) = \bar{p}_0 \dfrac{d^2}{dx^2} + \bar{p}_1 \dfrac{d}{dx} + w(x)\, p_2(x)$

where $\quad \bar{p}_0 = \exp\left(\displaystyle\int \frac{p_1}{p_0}\,dx\right)$ and $\bar{p}_1 = \dfrac{p_1}{p_0}\exp\left(\displaystyle\int \frac{p_1}{p_0}\,dx\right)$

b. Since $\quad \bar{p}_0{}' = \bar{p}_1$, then $\quad \underline{w(x)\mathcal{L}(x) \text{ is self-adjoint !}}$

3. <u>Scalar Product</u>: $\langle f | w\mathcal{L}g \rangle = \displaystyle\int_a^b f^*(x)\, w(x)\,\mathcal{L}g(x)\,dx$

a. We can show this becomes $\quad = \left[f^*\,\bar{p}_0 g' - f^{*\prime}\,\bar{p}_0 g \right]_a^b + \underbrace{\displaystyle\int_a^b w(x)\,\mathcal{L}f^*(x)\, g(x)\,dx}_{= \langle w\mathcal{L}f | g\rangle}$

b. But, this is equivalent to

$$\langle f | \mathcal{L}g \rangle = \langle \mathcal{L}f | g \rangle \quad \text{for} \quad \langle f | g\rangle = \int_a^b f^*(x)\, g(x)\, \underset{\underset{\text{weight function}}{\uparrow}}{w(x)}\,dx$$

c. Can show eigenfunctions are <u>orthogonal</u> !

4. <u>Summary</u>:

a. Self-adjoint $\mathcal{L}$ ($p_0' = p_1$), then it is Hermitian if
　i. Scalar product has uniform weight
　ii. Boundary conditions remove end points.

b. Non-self adjoint $\mathcal{L}$ ($p_0' \neq p_1$) can be made Hermitian if
　i. Scalar product includes appropriate weight factor $w(x)$
　ii. Boundary conditions remove endpoints

5. Ex. <u>Laguerre Functions</u>: $\mathcal{L} = x\dfrac{d^2}{dx^2} + (1-x)\dfrac{d}{dx}$ for $0 \leq x < \infty$. and $\psi(\infty) = 0$.

a. $p_0' = 1 \neq p_1$, so not self-adjoint.

b. $w(x) = p_0^{-1}\exp\left[\displaystyle\int \frac{p_1}{p_0}\,dx\right] = \dfrac{1}{x}\exp\left[\displaystyle\int \frac{1-x}{x}\,dx\right] = \dfrac{1}{x}\exp\left[\ln x - x\right] = \dfrac{1}{x}x\,e^{-x}$

$$\boxed{w(x) = e^{-x}}$$

c. Thus $\langle f | g\rangle = \displaystyle\int_0^\infty f^*(x)\, g(x)\, e^{-x}\,dx \quad \Big\}$

d. B.C's: $\left[x e^{-x}\left(f^* g' - f^{*\prime} g\right)\right]_0^\infty = 0 \quad \Big\}$ <u>Hermitian!</u>

IV. D. ODE Eigenvalue Problem: Legendre Equation

1. $\mathcal{L}\, y(x) = -(1-x^2)y'' + 2x\,y' = \lambda y \qquad -1 \le x \le 1$

2a. Note: ODE has regular singular points at $x = \pm 1$.

  b. Requirement that y is nonsingular at $x = \pm 1 \Rightarrow$ <u>Boundary Conditions!</u>

3. Frobenius' Method! $y = \sum_{j=0}^{\infty} a_j\, x^{s+j}$

  a. Indicial equation $s(s-1) = 0 \Rightarrow \boxed{s=0}, \boxed{s=1}$

  b. $\boxed{s=0}$ solution i. $a_1 = 0$ can be chosen

      ii. Recurrence relation $a_{j+2} = \dfrac{j(j+1) - \overset{\text{eigenvalue}}{\lambda}}{(j+1)(j+2)}\, a_j$

  c. $\boxed{s=1}$ solution i. $a_1 = 0$

      ii. Recurrence relation $a_{j+2} = \dfrac{(j+1)(j+2) - \lambda}{(j+2)(j+3)}\, a_j$

4. <u>Examine Convergence:</u>

  a. $s=0$ solution: Ratio test $\lim\limits_{j \to \infty} \dfrac{u_{j+2}}{u_j} = \lim\limits_{j \to \infty} \dfrac{j(j+1) - \lambda}{(j+1)(j+2)} \dfrac{x^{j+2}}{x^j} = x^2 \le 1$ on $-1 \le x \le 1$.

  b. Thus converges for $|x| < 1$, but is indeterminate for $|x| = 1$.

  c. The more <u>sensitive</u> Gauss test proves <u>divergence for $|x| = 1$</u>.

5. Eliminate divergence by truncating infinite series:

  a. Choosing $\lambda = \ell(\ell+1)$ truncates series at $a_{\ell+2} = 0$!

  b. Thus, solutions with $\boxed{\lambda = \ell(\ell+1)}$ converge over $-1 \le x \le 1$.

  c. Solutions with $\lambda = \ell(\ell+1)$ are polynomials of degree ℓ.

6. $s=0$ solutions y_ℓ when <u>ℓ is even</u>, $s=1$ solutions y_ℓ with <u>ℓ odd</u>

7a. The ODE operator $\mathcal{L}$ is <u>self-adjoint</u>.

  b. Coefficient $p_0 = -(1-x^2)$ vanishes at $|x| = 1$, so boundary terms $\left[f^* p_0 g' - f^{*\prime} p_0 g \right]_{-1}^{1} = 0 \Rightarrow \boxed{\mathcal{L} \text{ is Hermitian}}$