Lecture #10: Hankel Functions, Modified Bessel Functions, and Asymptotic Expansion

I. Hankel Functions

A. Properties and Definitions

1. Useful for problems involving the propagation of spherical or cylindrical waves.

2. Definition:
$$\boxed{\begin{aligned} H_\nu^{(1)}(x) &= J_\nu(x) + i\,Y_\nu(x) \\ H_\nu^{(2)}(x) &= J_\nu(x) - i\,Y_\nu(x) \end{aligned}}$$
Linear combination of J_ν & Y_ν

   a. Analogous to $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$.

   b. For real arguments x, $H_\nu^{(1)}(x) = H_\nu^{(2)*}(x)$

3. Series Expansions:
Top (1)
Bottom (2)
$\gamma \equiv$ Euler-Mascheroni const.
$$\boxed{\begin{aligned} H_0(x) &= \pm i\frac{2}{\pi}\ln x + 1 \pm i\frac{2}{\pi}(\gamma - \ln 2) + \ldots \\ H_1(x) &= \mp i\,\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu + \ldots, \quad \nu > 0 \end{aligned}}$$

4. Recurrence Relations: Since linear combinations of J_ν, Y_ν, same recurrence
$$\boxed{\begin{aligned} H_{\nu-1}(x) + H_{\nu+1}(x) &= \frac{2\nu}{x}H_\nu(x) \\ H_{\nu-1}(x) - H_{\nu+1}(x) &= 2H_\nu'(x) \end{aligned}}$$

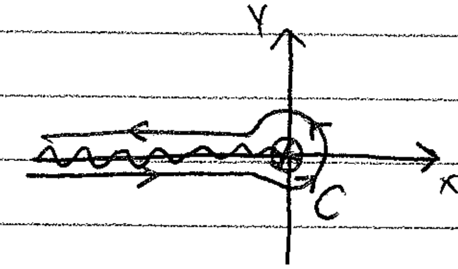
5. Wronskian Formulas:

   a. $$H_\nu^{(2)}H_{\nu+1}^{(1)} - H_\nu^{(1)}H_{\nu+1}^{(2)} = \frac{4}{i\pi x}$$

   b. $$J_{\nu-1}H_\nu - J_\nu H_{\nu-1} = \pm\frac{2}{i\pi x} \begin{cases} \text{top (1)} \\ \text{bottom (2)} \end{cases}$$

B. Contour Integral Representation

1. Recall
$$J_\nu(x) = \frac{1}{2\pi i}\int_C \frac{e^{\frac{x}{2}\left(t - \frac{1}{t}\right)}}{t^{\nu+1}}\,dt$$
where



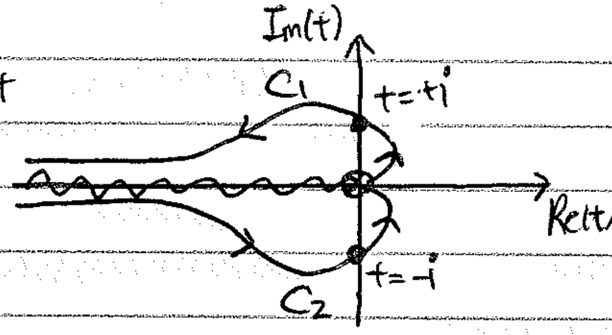
2. a. Bessel's ODE is satisfied for $\frac{e^{\frac{x}{2}(t-\frac{1}{t})}}{t^{\nu}}\left[\nu+\frac{x}{2}\left(t+\frac{1}{t}\right)\right]$ for any open contour C if this expression vanishes at the end points. For C above, vanishes at $t=-\infty$

b. But, this expression also vanishes for $\lim t\to 0^+$.

$$\lim_{t\to 0^+}\frac{e^{\frac{x}{2}(t-\frac{1}{t})}}{t^{\nu}}\left[\cancel{\nu}+\frac{x}{2}\left(\cancel{t}+\frac{1}{t}\right)\right]=\lim_{t\to 0^+}\frac{e^{-\frac{x}{2t}}}{t^{\nu+1}}\left(\frac{x}{2}\right)=0$$ since $e^{-\frac{x}{2t}}$ dominates as $t\to 0^+$

3. $$H_\nu^{(1)}(x)=\frac{1}{\pi i}\int_{C_1}\frac{e^{\frac{x}{2}(t-\frac{1}{t})}}{t^{\nu+1}}\,dt$$

$$H_\nu^{(2)}(x)=\frac{1}{\pi i}\int_{C_2}\frac{e^{\frac{x}{2}(t-\frac{1}{t})}}{t^{\nu+1}}\,dt$$



4. Use Method of Steepest Descents to evaluate

a. $$f(x)=\int_C g(x,t)F(x,t)\,dt\approx g(x,t_0)\int_C F(x,t)\,dt$$ where $F(x,t)=e^{\frac{x}{2}(t-\frac{1}{t})}$, $g(x,t)=\frac{1}{t^{\nu+1}}$

b. $$W(x,t)=\frac{x}{2}\left(t-\frac{1}{t}\right)$$

$$\frac{\partial W}{\partial t}=\frac{x}{2}\left(1+\frac{1}{t^2}\right)=0 \Rightarrow t=\pm i \text{ (saddle points)}$$

5. Why are integral representations so valuable

a. Can be used to develop relations among special functions.

b. Most important, can be used <u>to develop useful asymptotic expansions</u>!

II. Modified Bessel Functions: $I_\nu(x)$ and $K_\nu(x)$

A. Definition and Properties

1. <u>Modified Bessel ODE</u>: $$\rho^2\frac{d^2}{d\rho^2}P_\nu(k\rho)+\rho\frac{d}{d\rho}P_\nu(k\rho)-(k^2\rho^2+\nu^2)P_\nu(k\rho)=0$$ sign of this term changed!

a. Arises when constant k^2 in separation of variables has opposite sign.

b. Bessel ODE solutions $J_\nu(k\rho)$ & $Y_\nu(k\rho)$ are <u>oscillatory</u>

$\Rightarrow$ Modified Bessel ODE Solutions $I_\nu(k\rho)$ & $K_\nu(k\rho)$ are <u>exponential</u>

II. A. (Continued)

2. Substitution $k \to ik$ converts Bessel ODE to modified Bessel ODE.
   a. Thus Modified Bessel Functions are Bessel functions of imaginary argument

3. Series Solution

a. $$\boxed{I_\nu(x) = i^{-\nu} J_\nu(ix) = \sum_{s=0}^{\infty} \frac{1}{s!\,\Gamma(s+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2s}}$$

b. For small x, $I_\nu(x) \simeq \dfrac{x^\nu}{2^\nu \Gamma(\nu+1)} + \ldots$

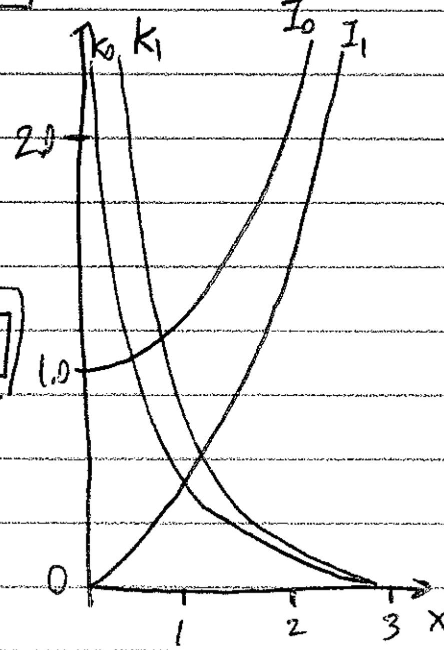
4. a. $I_\nu(x)$ and $I_{-\nu}(x)$ are linearly independent unless $\nu = n$ (integer)
   b. For integer n, $\boxed{I_n(x) = I_{-n}(x)}$

5. Recurrence Relations: Using $J_\nu(ix) = i^\nu I_\nu(x)$, we can derive

a. $$I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x} I_\nu(x)$$

b. $$I_{\nu-1}(x) + I_{\nu+1}(x) = 2 I'_\nu(x)$$

signs here swapped



B. Second Solution, $K_\nu(x)$

1. $$\boxed{K_\nu(x) \equiv \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) = \frac{\pi}{2} i^{\nu+1} \left[J_\nu(ix) + i Y_\nu(ix)\right]}$$

2. Thus, $$\boxed{K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi}}$$

3. Note that definition of $K_\nu(x)$ is not standardized!

4. $I_\nu(x)$ regular at $x=0$
   $K_\nu(x)$ irregular at $x=0$ $\Big\}$ see plot.

II. B. (Continued)

5. Recurrence:
$$K_{\nu-1}(x) - K_{\nu+1}(x) = \frac{-2\nu}{x} K_\nu(x)$$
$$K_{\nu-1}(x) + K_{\nu+1}(x) = -2K_\nu'(x)$$

6. For small x,
$$K_0(x) = -\ln x - \gamma + \ln 2 + \ldots$$
$$K_1(x) = 2^{\nu-1}\Gamma(\nu)x^{-\nu} + \ldots$$

C. Integral Representations

1. $$I_0(x) = \frac{1}{\pi}\int_0^\pi \cosh(x\cos\theta)\,d\theta$$

2. $$K_0(x) = \int_0^\infty \cos(x\sinh t)\,dt = \int_0^\infty \frac{\cos(xt)\,dt}{(1+t^2)^{\frac{1}{2}}} \quad , x>0$$

3. These are useful in developing asymptotic formulations

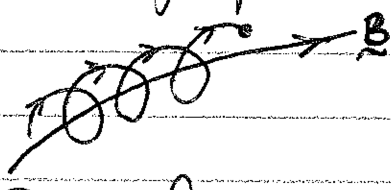
D. Uses of Modified Bessel Functions

1. $K_\nu(x)$ is useful in determining asymptotic behavior of all Bessel Fncs.

2. Modified Bessel Equation arises in a number of physical problems.

3. Can be used to simplify integrals over products of J_n:

a. Ex: In plasma physics, gyrokinetics averages over fast gyro-motion of charged particles in a magnetic field



b. Gyroaveraging of plane wave: $$\langle \phi(\underline{r},t)\rangle_{\underline{R}} = \frac{\hat{\phi}e^{i(\underline{k}\cdot\underline{R}-\omega t)}}{2\pi}\oint d\theta\, e^{i\frac{k_\perp v_\perp}{\Omega}\cos\theta}$$
$$= J_0\left(\frac{k_\perp v_\perp}{\Omega}\right)\hat{\phi}e^{i(\underline{k}\cdot\underline{R}-\omega t)}$$

c. Computing fields requires
$$\int_0^\infty dx\; x J_n(px)J_n(qx)e^{-a^2x^2} = \frac{1}{2a^2}I_n\left(\frac{pq}{2a^2}\right)e^{-\frac{(p^2+q^2)}{4a^2}}$$

Integrals over J_n^2 yield I_n!

III. Asymptotic Expansions

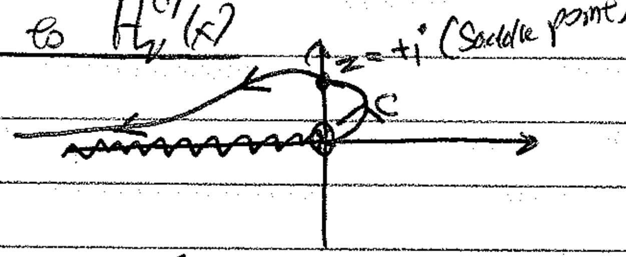
A. Basic Concepts

1. How does a Bessel function behave for large argument?
   a. Computers are not very helpful → Power series expansions converge slowly

2. We want to relate asymptotic series for $J_\nu(x)$ for $x \gg 1$ to power series expansion for $J_\nu(x)$ for $x \ll 1$.
   a. One approach is to use integral representation and apply the method of steepest descent.

B. Apply Method of Steepest Descent to $H_\nu^{(1)}(x)$

1. a. $H_\nu^{(1)}(t) = \frac{1}{\pi i}\int_C e^{\frac{t}{2}(z-\frac{1}{z})}\frac{dz}{z^{\nu+1}}$



b. Want asymptotic form for positive $t \gg 1$.

2. Method of Steepest Descent: $\int_C g(z,t)e^{w(z,t)}dz \simeq g(z_0,t)e^{w(z_0,t)}e^{i\theta}\sqrt{\frac{2\pi}{|w_0''|}}$

a. $g(z,t) = z^{-(\nu+1)}$

b. $w(z,t) = \frac{t}{2}(z-\frac{1}{z})$

$w' = \frac{\partial w}{\partial z} = \frac{t}{2}(1+\frac{1}{z^2}) = 0 \Rightarrow z^2 = -1 \Rightarrow \boxed{z_0 = +i}$ is a saddle point for $H_\nu^{(1)}(t)$

$w'' = -\frac{t}{z^3}$

c. Thus $w_0'' = w''(z_0,t) = \frac{-t}{(i)^3} = \frac{+t}{+i} = -it = te^{-i\frac{\pi}{2}}$ $\boxed{w_0'' = -it}$

d. So $\arg(w_0'') = -\frac{\pi}{2}$

e. $\theta = -\frac{\arg(w_0'')}{2} + (\frac{\pi}{2} \text{ or } \frac{3\pi}{2}) = +\frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4} \Rightarrow \boxed{\theta = \frac{3\pi}{4}}$

Choose $\frac{\pi}{2}$ to be consistent with contour C above

III. B2 (Continued)

f. Thus $H_\nu^{(1)}(t) = \frac{1}{\pi i}(ti)^{-\nu-1} e^{\frac{t}{2}(i-\frac{1}{i})} e^{i\frac{3\pi}{4}} \sqrt{\frac{2\pi}{t}}$

$$= \frac{1}{\pi} i^{-\nu-2} e^{\frac{t}{2}2i} e^{i\frac{3\pi}{4}} \sqrt{\frac{2\pi}{t}} = \sqrt{\frac{2}{\pi t}}\, e^{i\frac{\pi}{2}(-\nu-2)} e^{it} e^{i\frac{3\pi}{4}}$$

g. exponent: $i\left[-\frac{\nu\pi}{2} - \pi + t + \frac{3\pi}{4}\right] = i\left[t - \frac{\nu\pi}{2} - \frac{\pi}{4}\right]$

3. Thus, leading order term of asymptotic expansion is

a. $$\boxed{H_\nu^{(1)}(t) \approx e^{i(t-\frac{\nu\pi}{2}-\frac{\pi}{4})}\sqrt{\frac{2}{\pi t}}}$$

b. Also $$\boxed{H_\nu^{(2)}(t) \approx H_\nu^{(1)*}(t) = e^{-i(t-\frac{\nu\pi}{2}-\frac{\pi}{4})}\sqrt{\frac{2}{\pi t}}}$$

4. Using $K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix)$ yields (for $t = ix$)

$$\boxed{K_\nu(x) \approx \sqrt{\frac{\pi}{2x}}\, e^{-x}}$$

C. Expanding Integral Representation of $K_\nu(x)$ to Obtain Asymptotic Series

1. Consider the integral form $$\boxed{K_\nu(z) = \frac{\pi^{\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})}\left(\frac{z}{2}\right)^\nu \int_1^\infty e^{-zx}(x^2-1)^{\nu-\frac{1}{2}}dx}, \quad \nu > -\frac{1}{2} \;\&\; Re(z) > 0$$

2. To verify this integral representation, the form above must:

I. Satisfy the Modified Bessel ODE

II. Reproduce correct behavior at $z \ll 1$

III. Reproduce correct behavior at $z \gg 1$

Since modified Bessel ODE (2nd order) has two solutions, and we know
i) $I_\nu(x)$ is finite at $x=0$, diverges at $x=\infty$
ii) $K_\nu(x)$ is finite at $x=\infty$, diverges at $x=0$
$\Rightarrow$ Sufficient to verify form above.

3. To verify behavior at $z \ll 1$, use $\boxed{x = 1 + \frac{t}{z}}$ Change dx to dt
a. Changes $\int_1^\infty$ to $\int_0^\infty$
b. Isolates e^{-z} dependence.

4. Verifying behavior at $z \gg 1$ also yields asymptotic series
a. After $x = 1 + \frac{t}{z}$ transformation, integral may be manipulated to
$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} \frac{e^{-z}}{\Gamma(\nu+\frac{1}{2})} \int_0^\infty e^{-t} t^{\nu-\frac{1}{2}} \left(1 + \frac{t}{2z}\right)^{\nu-\frac{1}{2}} dt$$
b. Use binomial theorem to expand $\left(1 + \frac{t}{2z}\right)^{\nu-\frac{1}{2}}$ at $z \gg 1$
c. Exchange sum and integration and perform integration
d. Rearrangement of the result ultimately yields the asymptotic series
$$\boxed{K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[P_\nu(iz) + i Q_\nu(iz)\right]}$$
where
$$P_\nu(z) \sim 1 - \frac{(\mu-1)(\mu-9)}{2!(8z)^2} + \frac{(\mu-1)(\mu-9)(\mu-25)(\mu-49)}{4!(8z)^4} - \cdots$$
and
$$Q_\nu(z) \sim \frac{\mu-1}{1!(8z)} - \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3} + \cdots \quad \text{with } \mu = 4\nu^2$$

5. Key Points: Asymptotic Series
a. As an infinite series, $K_\nu(z)$ above is divergent!
★★★ b. But, for sufficiently large z, $K_\nu(z)$ may be approximated to any fixed degree of accuracy with a small number of terms.
c. Asymptotic (rather than convergent) character arises because binomial expansion is valid only for $t < 2z$, but we integrated t to infinity (e^{-t} term prevents disaster!)

III. D. Asymptotic Forms of All Bessel Functions

1. Hankel:
$$\boxed{H_\nu(z)=\sqrt{\frac{2}{\pi z}}\,e^{\pm i\left[z-(\nu+\frac{1}{2})\frac{\pi}{2}\right]}\left[P_\nu(z)\pm i\,Q_\nu(z)\right]}$$
top (1)
bottom (2)

2. Bessel:
$$J_\nu(z)=\sqrt{\frac{2}{\pi z}}\left\{P_\nu(z)\cos\left[z-(\nu+\tfrac{1}{2})\tfrac{\pi}{2}\right]-Q_\nu(z)\sin\left[z-(\nu+\tfrac{1}{2})\tfrac{\pi}{2}\right]\right\}$$

3. Neumann:
$$Y_\nu(z)=\sqrt{\frac{2}{\pi z}}\left\{P_\nu(z)\sin\left[z-(\nu+\tfrac{1}{2})\tfrac{\pi}{2}\right]+Q_\nu(z)\cos\left[z-(\nu+\tfrac{1}{2})\tfrac{\pi}{2}\right]\right\}$$

4. Modified First:
$$\boxed{I_\nu(z)=\frac{e^z}{\sqrt{2\pi z}}\left[P_\nu(iz)-i\,Q_\nu(iz)\right]}$$

E. Properties

1. At $z\gg 1$, $P_\nu\to 1$, $Q_\nu\to\frac{1}{z}$

2. All Bessel functions have leading asymptotic terms $\propto\frac{1}{z^{\frac{1}{2}}}$ multiplied by either a real or complex exponential
exponential increase or decrease ↗ real; complex ← oscillatory

3. Multiplied by $e^{\pm i\omega t}$, Hankel functions describe traveling waves ("in" or "out")

4a. For $z\gg 1$, J_ν, Y_ν, H_ν are phase-shifted sinusoidal functions
b. Phase shift, $-(\nu+\frac{1}{2})\frac{\pi}{2}$ is the same for same order ν.

5. $I_\nu(z)$ regular at $z=0$, $K_\nu(z)$ regular at $z=\infty$.

6. For half-integral ν, $P_\nu(z)$ & $Q_\nu(z)$ series terminate as polynomials.
$\Rightarrow$ exact!

7. Accuracy: For $J_n(x)\sim\sqrt{\frac{2}{\pi x}}\cos\left[x-(n+\frac{1}{2})\frac{\pi}{2}\right]$ to be accurate, need sine term to be negligible $\Rightarrow Q_n(z)\sim\frac{4n^2-1}{8x}\ll 1 \Rightarrow \boxed{8x\gg 4n^2-1}$

F. Ex: Cylindrical Traveling Waves

1. 2D wave problem: Source at $r=0$, azimuthal symmetry $\Rightarrow$ $m=0$
a. For large r, we want $U\sim e^{i(kr-\omega t)}$

2. Asymptotically $(r\gg 1)$, we have $U(r,t)\sim\underbrace{H_0^{(1)}(kr)}_{\text{Hankel Function!}}e^{-i\omega t}\sim e^{i(kr-\omega t)}$