

Lecture #B: Associated Legendre Functions, Spherical Harmonics, and Second Kind

I. Associated Legendre Functions

A. Parity

1.  $P_l^m(-x) = (-1)^{l+m} P_l^m(x)$  (Parity of  $l+m$ )

2. Values at endpoints  $x = \pm 1$ :

a. Since  $P_l^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_l(x)$ ,

$P_l^m(\pm 1) = 0$  for  $m \neq 0$

b. For  $m=0$ ,  $P_l^0(x) = P_l(x)$ , so we recover  $P_l(1) = 1$ ,  $P_l(-1) = (-1)^l$

3. Value of  $P_l^m(0)$ :

$$P_l^m(0) = \begin{cases} (-1)^{\frac{l+m}{2}} \frac{(l+m-1)!!}{(l-m)!!} & l+m \text{ even} \\ 0 & l+m \text{ odd} \end{cases}$$

Recall:  $(2n)!! = 2^n n!$  and  $(2n-1)!! = \frac{(2n)!}{2^n n!}$

B. Orthogonality

1. For each  $m$ , all  $P_l^m(x)$  for  $l > |m|$  are orthogonal!

a. NOTE: Recall for each  $l$ , there are  $2l+1$  possible  $m$  values,  $-l \leq m \leq l$ .

2. Orthogonality Relation of  $l$  &  $l'$ :

$$\int_0^\pi P_p^m(\cos\theta) P_q^m(\cos\theta) \sin\theta d\theta = \int_{-1}^1 P_p^m(x) P_q^m(x) dx = \frac{2}{2p+1} \frac{(p+m)!}{(p-m)!} \delta_{pq}$$

a. Intends of  $P_l^m(x)$ , where  $P_l^m(x) = (1-x^2)^{\frac{m}{2}} P_l^m(x)$ ,

$$\int_{-1}^1 P_p^m(x) P_q^m(x) (1-x^2)^m dx = \frac{2}{2p+1} \frac{(p+m)!}{(p-m)!} \delta_{pq}$$

b.  $\Rightarrow$  polynomials  $P_l^m(x)$  are orthogonal with weight  $(1-x^2)^m$

# I. B. 2. (Continued)

c. In contrast,  $P_l^m(x)$  are orthogonal with unit weight.

## 3. Orthogonality Relation of $m$ & $m'$

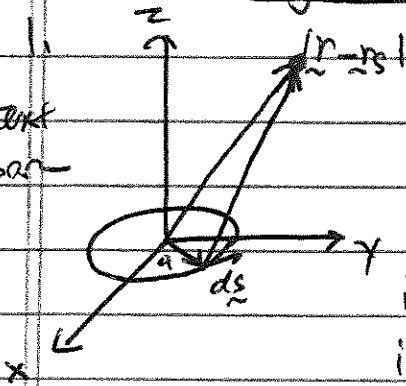
a. 
$$\int_{-1}^1 P_l^m(x) P_l^{m'}(x) (1-x^2)^{-1} dx = \frac{(l+m)!}{m!(l-m)!} \delta_{mm'}$$

b. Note really very useful:  $\Phi(\phi) = a_m e^{\pm im\phi}$  is already orthogonal in  $m$ , so we don't generally need to deal with the orthogonality of  $P_l^m(\cos\theta)$  with respect to  $m$ !

## C. Ex: Magnetic Field from a Circular Current Loop

NOTE:

Figure in text is very poor - loop is not tilted!



2. 
$$d\underline{A}(r) = \frac{\mu_0}{4\pi} \frac{I ds}{|r-rs|}$$

a. By symmetry,  $\underline{A} = A_\phi(r, \theta) \hat{e}_\phi$

i) Axisymmetry  $\Rightarrow$  cannot depend on  $\phi$ .

ii)  $d\underline{A}$  is in direction of  $ds$  (in plane of constant  $z$ )

iii) In spherical coord's, 
$$\nabla \cdot \underline{A} = \frac{1}{r^2 \sin\theta} \left[ \sin\theta \frac{\partial}{\partial r} (r^2 A_r) + r \frac{\partial}{\partial \theta} (\sin\theta A_\theta) + r \frac{\partial A_\phi}{\partial \phi} \right]$$

ZA  $A_r \neq 0$  or  $A_\theta \neq 0$ , then  $\nabla \cdot \underline{A}$  would diverge at  $r=0$ .

3. Ampere-Maxwell Law: 
$$\nabla \times \underline{B} = \mu_0 \underline{j} + \epsilon_0 \mu_0 \frac{\partial \underline{E}}{\partial t} \Rightarrow \nabla \times \underline{B} = 0$$
  
no current except on loop      static

4. a.  $\underline{B} = \nabla \times \underline{A}$

b. 
$$\nabla \times [\nabla \times A_\phi(r, \theta) \hat{e}_\phi] = 0$$

$$\Rightarrow \left[ \frac{\partial^2 A_\phi}{\partial r^2} + \frac{2}{r} \frac{\partial A_\phi}{\partial r} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left( \sin\theta \frac{\partial A_\phi}{\partial \theta} \right) - \frac{1}{r^2 \sin^2\theta} A_\phi \right] = 0$$

5. By Usual Separation of Variables,  $A_\phi(r, \theta) = R(r) \Theta(\theta)$ .

# I. C. 5. (Continued)

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a.  $r^2 \frac{d^2 R}{dr^2} + 2 \frac{dR}{dr} - l(l+1)R = 0$

b.  $\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + l(l+1)\Theta - \frac{\Theta}{\sin^2 \theta} = 0$  Associated Legendre Eq with  $m=1$

c. General Solution:  $A_\phi(r, \theta) = \sum_{l=0}^{\infty} \left[ c_l \left(\frac{a}{r}\right)^{l+1} + d_l \left(\frac{r}{a}\right)^l \right] P_l^1(\cos \theta)$   
 want solution for  $r > a$ , so  $d_l = 0$ .

6. Solution for Components of  $\vec{B}$ :

a.  $B_r(r, \theta) = (\nabla \times A_\phi \hat{e}_\phi)_r = \frac{\cot \theta}{r} A_\phi + \frac{1}{r} \frac{\partial A_\phi}{\partial \theta}$

b.  $B_\theta(r, \theta) = (\nabla \times A_\phi \hat{e}_\phi)_\theta = -\frac{1}{r} \frac{\partial (r A_\phi)}{\partial r}$

7. To evaluate  $\theta$  derivative, use derivative relation

$$(1-x^2)^{\frac{1}{2}} [P_l^m(x)]' = (l-m)(l-m+1) P_l^{m-1} + \frac{m x}{(1-x^2)^{\frac{1}{2}}} P_l^m(x)$$

with  $m=1$  and  $x = \cos \theta$  to obtain

$$\frac{d P_l^1(\cos \theta)}{d\theta} = -\sin \theta \frac{d P_l^1(\cos \theta)}{d(\cos \theta)} = -l(l+1) P_l(\cos \theta) - \cot \theta P_l^1(\cos \theta)$$

Cancel's other  $\cot \theta$  term!

8. Thus

a.  $B_r(r, \theta) = -\frac{1}{r} \sum_{l=1}^{\infty} l(l+1) c_l \left(\frac{a}{r}\right)^{l+1} P_l(\cos \theta)$

b.  $B_\theta(r, \theta) = \frac{1}{r} \sum_{l=1}^{\infty} l c_l \left(\frac{a}{r}\right)^{l+1} P_l^1(\cos \theta)$

9. Now, we must determine coefficients  $c_l$  by matching solution

a. Choose polar axis, where we may calculate  $B_r = B_z$ .

b. Along  $\theta=0$ ,

$$B_r(z, 0) = -\frac{a^2}{z^3} \sum_{s=0}^{\infty} (s+1)(s+2) c_{s+1} \left(\frac{a}{z}\right)^s$$

Z.C. (Continued)

10. Along polar axis, Biot-Savart Law may be integrated to yield

$$dB = \frac{\mu_0 I}{4\pi r^2} ds \sin\theta$$

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$$B_z = \frac{\mu_0 I a^2}{2z^3} \left(1 + \frac{a^2}{z^2}\right)^{-\frac{3}{2}} = \frac{\mu_0 I a^2}{2z^3} \sum_{s=0}^{\infty} (-1)^s \frac{(2s+1)!!}{(2s)!!} \left(\frac{a}{z}\right)^{2s}$$

11. We thereby find

$$\begin{aligned} C_{2s} &= 0 \\ C_{2s+1} &= \frac{\mu_0 I}{2} (-1)^{s+1} \frac{(2s-1)!!}{(2s+2)!!} \end{aligned}$$

## II. Spherical Harmonics

### A. Generalized Angular $(\theta, \phi)$ Solutions

1. In spherically symmetric problems,  $H(\theta)\Phi(\phi)$  are always the same for Laplace, Helmholtz, and Schrödinger equations

2.  $\Phi$  Solution: For integer  $m$ ,

$$a. \Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \quad \text{or} \quad \Phi_m(\phi) = \frac{1}{\sqrt{\pi}} \begin{cases} 1/\sqrt{2} & m=0 \\ \cos m\phi & m>0 \\ \sin |m|\phi & m<0 \end{cases}$$

$$b. \int_0^{2\pi} [\Phi_m(\phi)]^* \Phi_{m'}(\phi) d\phi = \delta_{mm'} \quad \text{Orthogonality}$$

3.  $H$  Solution:

$$a. H_{lm}(\cos\theta) = \sqrt{\frac{2l+1}{2} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) \quad \text{for } -l \leq m \leq l$$

$$b. \text{Orthogonality} \int_0^\pi [H_{lm}(\cos\theta)]^* H_{l'm}(\cos\theta) \sin\theta d\theta = \delta_{ll'}$$

Same value of  $m$ !

4. Def: Spherical Harmonics

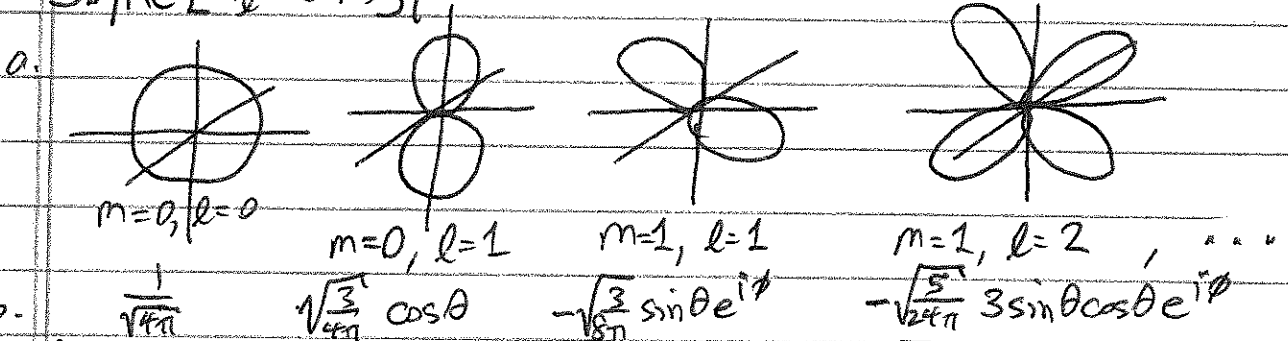
a.  $Y_l^m(\theta, \phi) \equiv \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi}$

b.  $\int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta [Y_l^m(\theta, \phi)]^* Y_{l'}^{m'}(\theta, \phi) = \delta_{ll'} \delta_{mm'}$

c. Normalized Solutions of Sturm-Liouville problem.

⇒ Complete set of orthonormal eigenfunctions over spherical surface

5.  $|\text{Re}[Y_l^m(\theta, \phi)]|^2$



6. Cartesian Representation: a.  $\cos\theta = \frac{z}{r}$   
 b.  $\sin\theta e^{\pm i\phi} = \frac{x}{r} \pm i \frac{y}{r}$

B. General Solutions:

1. Laplace Eq:  $\nabla^2 \psi = 0$   $\psi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l (a_{lm} r^l + b_{lm} r^{-l-1}) Y_l^m(\theta, \phi)$

2. Helmholtz Eq:  $\nabla^2 \psi + k^2 \psi = 0$   $\psi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l [a_{lm} j_l(kr) + b_{lm} y_l(kr)] Y_l^m(\theta, \phi)$

↑  
Spherical Bessel Functions

## II. Continued

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### C. Laplace Expansion

1. Thm Any function  $f(\theta, \phi)$  on a spherical surface may be expanded in  $Y_l^m(\theta, \phi)$

$$a. f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l C_{lm} Y_l^m(\theta, \phi)$$

$$b. \text{ where } C_{lm} = \langle Y_l^m | f(\theta, \phi) \rangle = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta Y_l^m(\theta, \phi) f(\theta, \phi)$$

2. Ex Electrostatic Potential within sphere with potential  $V(r_0, \theta, \phi)$

a. General Solution for  $r < r_0$  
$$V(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{lm} r^l Y_l^m(\theta, \phi)$$

b. Determine  $a_{lm}$  on sphere at  $r=r_0$ :

$$V(r_0, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l C_{lm} Y_l^m(\theta, \phi) \quad \text{where } C_{lm} = a_{lm} r_0^l$$

and 
$$C_{lm} = \langle Y_l^m | V(r_0, \theta, \phi) \rangle$$

c. Thus 
$$V(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l C_{lm} \left(\frac{r}{r_0}\right)^l Y_l^m(\theta, \phi) \quad \text{with } C_{lm} \text{ above}$$

### D. Symmetry

1. NOTE: A spherically symmetric problem can have solutions with less than full spherical symmetry

a. Ex: Earth orbits in a plane, but Sun's gravitational field is spherically symmetric  $\Rightarrow$  "initial conditions" determine orbital plane.

2. For a given  $l$ ,  $2l+1$  mutually orthogonal eigenfunctions,  $Y_l^m$  ( $-l \leq m \leq l$ )

b. Any solution (for fixed  $l$ ) is a linear combination of these eigenfunctions.

3. Consider angular solution for given  $l$  as belonging to a Hilbert space with  $(2l+1)$  members.

4. Coordinate system rotations cannot change  $r$  dependence, so functions  $Y_l^m$  ( $-l \leq m \leq l$ ) are closed under rotation.

### 5. Quantum Mechanics:

a. All solutions with same  $l$  have same Energy  $E$  and radial functions:  
 $\Rightarrow (2l+1)$ -fold degeneracy of eigenstates of given  $l$ .

6. Along polar axis: a.  $\theta=0 \Rightarrow Y_l^m(0, \phi) = \sqrt{\frac{2l+1}{4\pi}} \delta_{m0}$

b.  $\theta=\pi \Rightarrow Y_l^m(\pi, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi}} \delta_{m0}$

7. Note also that one may derive recurrence relations for  $Y_l^m(\theta, \phi)$ .

## III. Legendre Functions of the Second Kind

### A. Compute Second, Linearly Independent Solution

$$1. \quad Y'' - \underbrace{\frac{2x}{1-x^2}}_{=f_1(x)} Y' - \underbrace{\frac{l(l+1)}{1-x^2}}_{=f_2(x)} Y = 0$$

2. Using method of computing  $y_2(x)$  from  $y_1(x)$  (see Sec 7.6),

$$y_2(x) = y_1(x) \int \frac{\exp[-\int^x f_1(x_2) dx_2]}{[y_1(x_2)]^2} dx_2,$$

we obtain for  $y_1(x) = P_l(x)$ ,

$$Q_l(x) = P_l(x) \int \frac{dx}{(1-x^2)[P_l(x)]^2}$$

### III. A. (Continued)

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3. Thus,  $Q_0(x) = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right)$

$$Q_1(x) = \frac{x}{2} \ln \left( \frac{1+x}{1-x} \right) - 1$$

$$Q_2(x) = \frac{1}{2} P_2(x) \ln \left( \frac{1+x}{1-x} \right) - \frac{3x}{2}$$

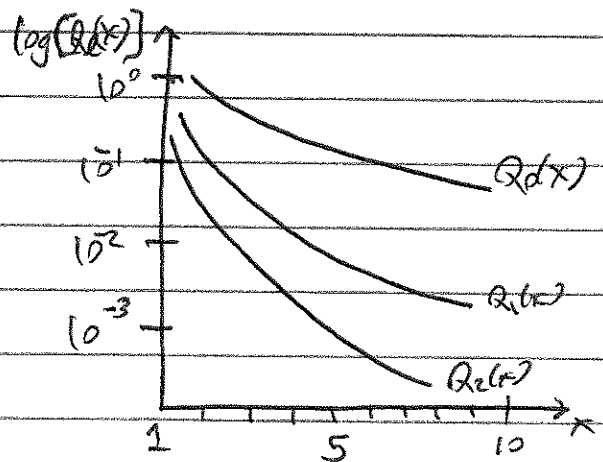
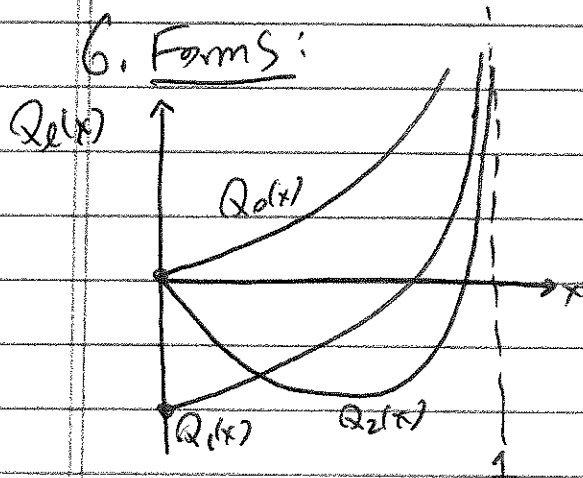
4. Must obey the same Recurrence Formulas for Legendre Equation.

b. May be used to determine formulas for higher  $l$

$$Q_l(x) = \frac{1}{2} P_l(x) \ln \left( \frac{1+x}{1-x} \right) - \frac{2l-1}{1 \cdot l} P_{l-1}(x) - \frac{2l-5}{3(l-1)} P_{l-3}(x) - \dots$$

5.  $Q_l(x)$  are often used in range  $x > 1$  (outside  $-1 < x < 1$ ),  
so you may replace  $\ln$  term with  $\ln \left( \frac{x+1}{x-1} \right)$

6. Forms:



B. Properties:

1.  $Q_l(x)$  is  $\begin{cases} \text{odd function} & \text{for even } l \\ \text{even function} & \text{for odd } l \end{cases}$

2.  $Q_l(1) = \infty$  for all  $l$

3.  $Q_l(0) = \begin{cases} 0 & \text{for even } l \\ (-1)^{s+1} \frac{(2s)!!}{(2s+1)!!} & \text{for odd } l \end{cases}$