Lecture #14: Fourier Series

I. Fourier Series

A. General Properties

1. Fourier Series are a valuable tool for solving ODEs or PDEs with periodic boundary conditions

2. Consider a piecewise regular function $f(x)$ on $[0, 2\pi]$,
   ↰ may contain finite discontinuities!

a. Define: Fourier Series

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}}$$

b. To determine complex coefficients c_n, multiply by e^{-inx} and $\int_0^{2\pi} dx$:

$$\int_0^{2\pi} f(x) e^{-inx}\, dx = \sum_{n'=-\infty}^{\infty} c_{n'} \int_0^{2\pi} e^{in'x} e^{-inx}\, dx$$

c. Note, $\int_0^{2\pi} e^{i(n'-n)x}\, dx = 2\pi \delta_{nn'}$

d. Thus $\quad = \sum_{n'=-\infty}^{\infty} c_{n'}\, 2\pi \delta_{nn'} = 2\pi c_n$

e. So

$$\boxed{c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx}\, dx}$$

3. Alternatively, we may define Fourier series by real functions $\cos nx$ & $\sin nx$:

a. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx, \quad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx, \quad n = 1, 2, 3, \ldots$$

I. A. (Continued)

4. The complex and real representation coefficients are related by

$$c_0 = \frac{a_0}{2} \qquad c_n = \frac{a_n - i b_n}{2}, \quad c_{-n} = \frac{a_n + i b_n}{2}, \quad n>0$$

B. Sturm-Liouville Theory

1. Recall that a Hermitian operator for a problem with periodic boundary conditions is a Sturm-Liouville Problem.
   Therefore: a) Eigenfunctions are real.
   b) Complete basis of orthogonal eigenfunctions.

2. For this problem
   a. ODE: $-y''(x) = \lambda y(x)$
   b. $\langle f|g\rangle = \int_0^{2\pi} f^*(x) g(x)\, dx$
   c. Boundary Conditions: $y(0) = y(2\pi)$, $y'(0) = y'(2\pi)$
   d. Eigenfunctions: $\phi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ $\quad n = \dots, -1, 0, 1, \dots$

   or $\phi_0 = \frac{1}{\sqrt{2\pi}}$ $\quad \phi_n = \frac{\cos nx}{\sqrt{\pi}}$, $\phi_{-n} = \frac{\sin nx}{\sqrt{\pi}}$ for $n = 1, 2, 3, \dots$
   e. Eigenvalues: n^2

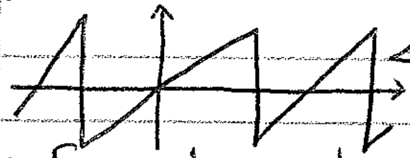
3. Normalization of eigenfunctions: Note!
   a) For complex representation, $\langle e^{inx}|e^{inx}\rangle = \int_0^{2\pi} e^{-inx} e^{inx}\, dx = 2\pi$
   b) ~~For~~ For real representation, $\int_0^{2\pi} \sin^2(nx)\, dx = \int_0^{2\pi} \cos^2(nx)\, dx = \pi$

C. Representing Discontinuous Functions

1. Power series cannot represent discontinuous functions
   a. Expansion about a point, with radius of convergence up to nearest singularity

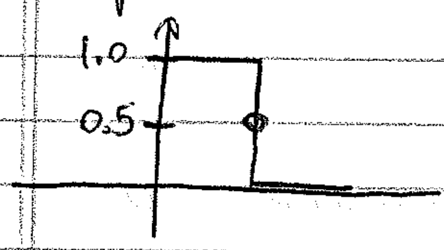
I. C. (Continued)

2. Fourier Series <u>can</u> describe functions with <u>finite</u> discontinuities.



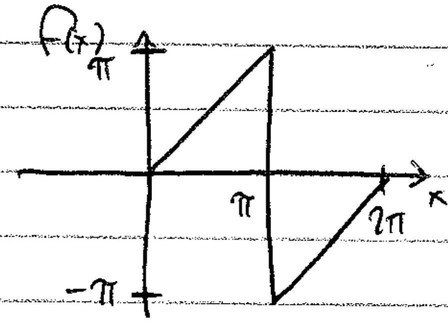
a. Expansion employs information over entire interval, so it can describe "nonpathological" singularities

3. For a <u>piecewise regular</u> function $f(x)$ (also known as satisfying <u>Dirichlet conditions</u>), Fourier series at a discontinuous point is limit of left and right approaches



$$f_{\text{Fourier Series}}(x_0) = \lim_{\epsilon \to 0} \left[\frac{f(x_0+\epsilon) + f(x_0-\epsilon)}{2} \right]$$

4. <u>Ex:</u> Sawtooth Wave

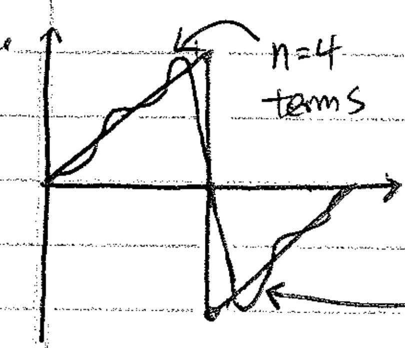


a. $f(x) = \begin{cases} x & 0 \le x < \pi \\ x - 2\pi & \pi < x \le 2\pi \end{cases}$

b. Using real representation, $a_n = 0$, $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx$ leads to

$$f(x) = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots + (-1)^{n+1} \frac{\sin nx}{n} + \ldots \right]$$

c.



d. Comments:

i. Accuracy of $f(x)$ increases with number of terms

ii. All curves pass through $f(\pi) = 0$

iii. Overshoot near discontinuity

$\Rightarrow$ <u>Gibbs phenomenon!</u>

I. (Continued)

D. <u>General Periodic Interval $[-L, L]$</u>

1. By taking interval ~~over~~ $[-\pi, \pi]$ instead of $[0, 2\pi]$ and substituting $y = x\frac{L}{\pi}$, we may obtain

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}} \qquad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}$$

or

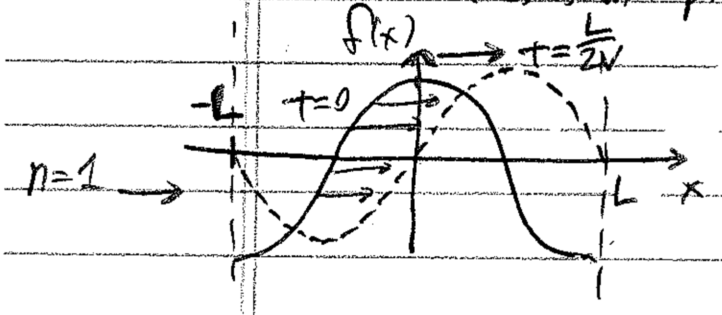
$$\boxed{c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\frac{n\pi x}{L}} dx} \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L} dx$$

E. <u>Traveling Waves</u>

1. A waveform may be traveling with phase velocity v, so we obtain terms $\sim \cos\left[\frac{n\pi}{L}(x - vt)\right]$

Where a constant phase of wave moves at $x - vt = \text{constant}$, or $\left(v = \frac{x}{t}\right)$



2. For a particular mode n,

a. <u>Wavelength</u> $\lambda_n = \frac{2L}{n}$

b. <u>Period</u> $T_n = \frac{\lambda_n}{v} = \frac{2L}{nv}$

c. <u>Linear Frequency</u>: $f = \frac{1}{T} = \frac{nv}{2L}$, <u>Angular Frequency</u> $\omega = \frac{2\pi}{T} = \frac{\pi n v}{L}$

d. <u>Fundamental Frequency</u>: $n = 1$

<u>Harmonic Frequencies</u>: $n > 1$

I. (Continued)

F. Linearity

1. Fourier Analysis is most useful for Linear Differential Equations

2a. For a linear problem, the general solution is a linear superposition of each Fourier component (each mode n)

b. Each Fourier component evolves independently of the others!

c. The solution for general mode n is applicable to all modes.

3a. For nonlinear problems, the nonlinear terms lead to coupling among different Fourier modes.

b. Numerically, efficient algorithms, such as the Fast Fourier Transform [Cooley & Tukey, (1965)], enable nonlinear evolution of Fourier components in time (pseudo spectral).

c. Pseudospectral algorithms evaluate linear terms in Fourier space and nonlinear terms in real space.

G. Symmetry

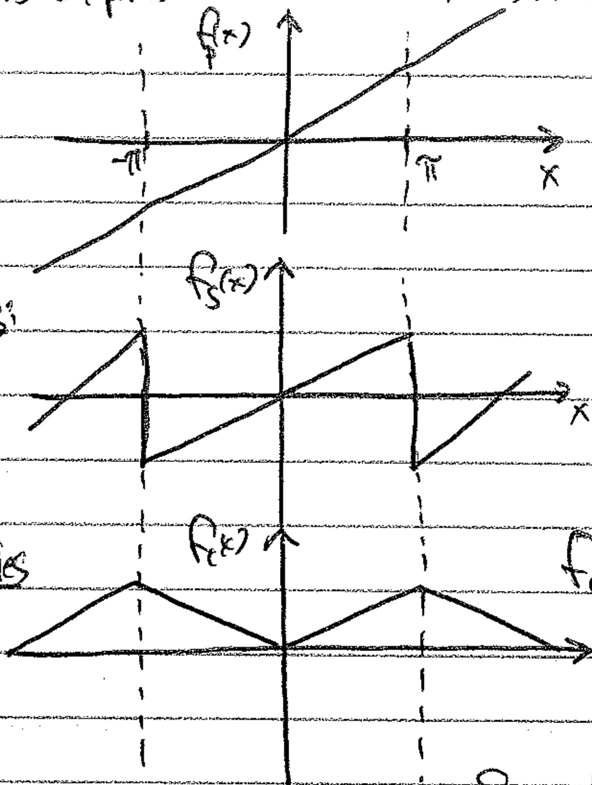
1. For functions $f(x)$ that have definite parity, the real representation of Fourier series ($\sin nx$ & $\cos nx$) leads to simplifications

a. For $f(x)$ even, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ $(b_n = 0)$

b. For $f(x)$ odd, $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ $(a_n = 0)$

2. Sometimes these are denoted Fourier cosine & Fourier sine series.

3. Ex: Various Expansions of $f(x)=x$ on $[0,\pi]$



a. Power Series:
Non-periodic

$f_p(x)=x$

b. Fourier Sine Series:
Odd parity

$$f_s(x)=2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$$

c. Fourier Cosine Series
Even parity

$$f_c(x)=\frac{\pi}{2}-\sum_{n=0}^{\infty}\frac{4}{\pi}\frac{\cos(2n+1)x}{(2n+1)^2}$$

Different outside of $[0,\pi]$ interval

Same form on $[0,\pi]$ for all three functions

H. Integration and Differentiation of Fourier Series

1. Consider term-by-term integration of $f(x)=\sum_{n=-\infty}^{\infty} c_n e^{inx}$

   a. $\int f(x)dx=\sum_{n=-\infty}^{\infty}\frac{c_n}{(in)}e^{inx}$ + constant

   b. Additional factor of n^{-1} in each term $\Rightarrow$ more rapid convergence!

2. Thus, $\boxed{\text{A convergent Fourier Series may always be integrated term by term}}$

3. Consider term-by-term differentiation of Sawtooth wave (a convergent Fourier series)

   a. $f(x)=x=2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(nx)}{n}$ on $-\pi\le x\le\pi$

   b. Differentiating: $f'(x)=1=2\sum_{n=1}^{\infty}(-1)^{n+1}\cos(nx) \leftarrow$ Not convergent!

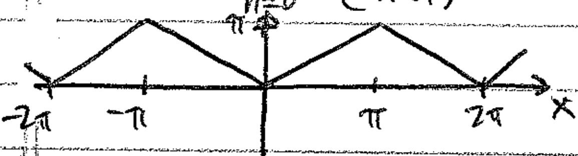
I. H. (Continued)

4a. $\boxed{\text{Term-by-term differentiation of a convergent Fourier Series } \underline{\text{may not}} \text{ lead to a convergent series for the derivative}}$

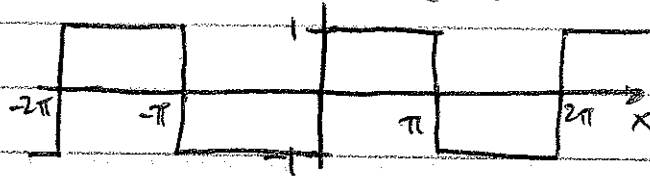
b. Always check the resulting series for convergence!

5. Ex: Derivative of triangular wave ⇒ Square wave

a. $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2}$



b. $f'(x) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{(2n+1)}$



6. Generally:

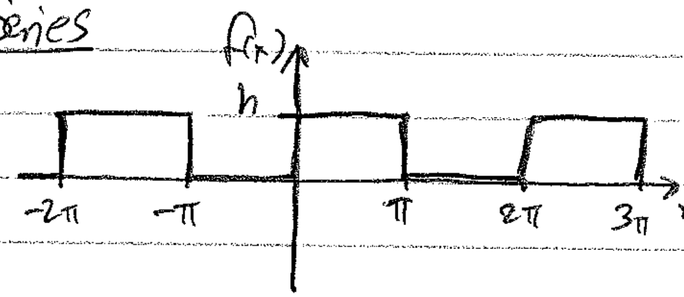
a. Differentiation reduces rate of convergence ⇒ may yield divergent series

b. Term-by-term differentiation usually permitted for a uniformly convergent series

II. Applications of Fourier Series

A. Example: Square Wave

1. $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ h & 0 < x < \pi \end{cases}$



2. NOTE: $f(x) - \frac{h}{2}$ is odd, so definite parity suggest real Fourier represantation is appropriate choice (even terms will be zero, except a_0)

3. $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$

a. For $n=0$, $a_0 = \frac{1}{\pi}\int_{-\pi}^{0} \cancel{(0)}dx + \frac{1}{\pi}\int_{0}^{\pi} h\,dx = \frac{h(\pi-0)}{\pi} = h$

b. For $n>0$, $a_n = \frac{1}{\pi}\int_{-\pi}^{0}\cancel{(0)\cos nx\,dx} + \frac{1}{\pi}\int_{0}^{\pi}(h)\cos nx\,dx = \frac{h}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} = \frac{h}{\pi}\frac{(\cancel{\sin n\pi} - \cancel{\sin 0})}{n}$

$= 0!$

II.A (Continued)

4. $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$

a. $b_n = \frac{1}{\pi}\int_{-\pi}^{0}(0)\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} h\sin(nx)\,dx = \frac{h}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = \frac{h}{\pi}\left[\frac{1-\cos n\pi}{n}\right]$

b. NOTE: $\cos n\pi = (-1)^n$, so $b_n = \frac{h}{\pi}\left[\frac{1-(-1)^n}{n}\right] = \begin{cases} 0, & n \text{ even} \\ \frac{2h}{n\pi}, & n \text{ odd}\end{cases}$

5. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$\boxed{f(x) = \frac{h}{2} + \frac{2h}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{(2n+1)}}$$

6. IMPORTANT NOTE:

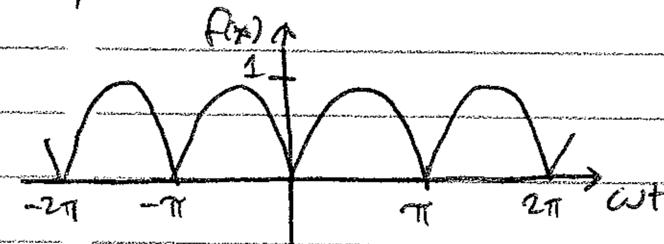
a. Coefficients decrease only as n^{-1} $\Rightarrow$ Convergence is slow!

b. Thus, high frequency harmonics do not drop off very fast

$\Rightarrow$ Signal contains significant contribution from high frequencies!

c. For numerical simulations using Fourier representation, resolution of discontinuities requires many modes $\Rightarrow$ not well suited!

B. Ex: Full Wave Rectifier



1. $f(t) = \begin{cases} \sin\omega t & 0 < \omega t < \pi \\ -\sin\omega t & -\pi < \omega t < 0 \end{cases}$

2. Even parity, so $\boxed{b_n = 0}$ for all n!

3.
a. $a_0 = \frac{1}{\pi}\int_{-\pi}^{0}(-\sin\omega t)\,d(\omega t) + \frac{1}{\pi}\int_0^{\pi}\sin\omega t\,d(\omega t) = \frac{2}{\pi}\int_0^{\pi}\sin\omega t\,d(\omega t) = \frac{4}{\pi}$

b. $a_n = \frac{2}{\pi}\int_0^{\pi}\sin\omega t\cos(n\omega t)\,d(\omega t) = \begin{cases} -\frac{2}{\pi}\frac{2}{n^2-1}, & n \text{ even} \\ 0, & n \text{ odd}\end{cases}$

4. $$\boxed{f(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n\omega t)}{(2n)^2-1}}$$ $\leftarrow$ coefficients decrease as n^{-2}!

C. Generally, 1. If $f(x)$ has discontinuities, $c_n \propto O(\frac{1}{n})$ $\Rightarrow$ Slow convergence!

2. If $f(x)$ continuous [but $f'(x)$ may be discontinuous], $c_n \propto O(\frac{1}{n^2})$ $\Rightarrow$ Faster convergence!