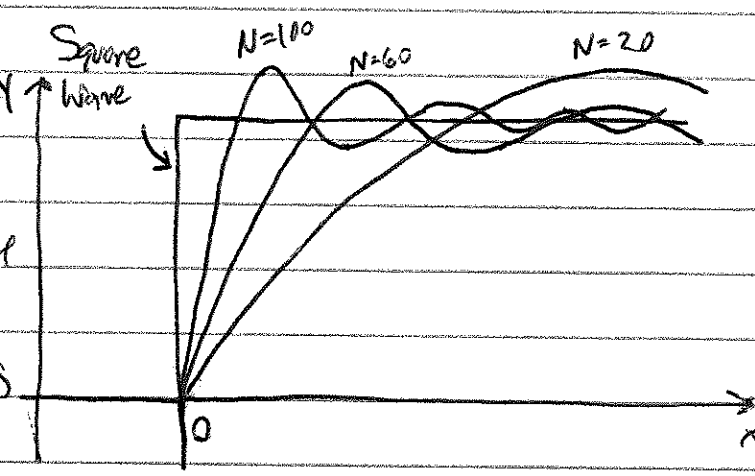
Lecture #15 Gibbs Phenomenon, Integral Transforms and Fourier Transforms

I. Gibbs Phenomenon

A. Basic Concept

1. When a Fourier Series with a finite number of terms N is used to represent a discontinuity (eg., a square wave), the result is a form with a significant overshoot!



2. This overshoot is called Gibbs phenomenon

3. Important implication for using a Fourier series to represent a system with discontinuities (eg, shocks!) in numerical simulations.

B. Quantitative Treatment: Partial sums

1. Consider a finite Fourier Series with terms $-N \leq n \leq N$, where $N \geq 0$,

a. $$f_N(x) = \sum_{n=-N}^{N} c_n e^{inx} \quad \text{where } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-int} dt$$

2. Combining these,
$$f_N(x) = \sum_{n=-N}^{N}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-int} dt\right] e^{inx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} dt\, f(t) \underbrace{\sum_{n=-N}^{N} e^{in(x-t)}}_{\text{Geometric series}}$$

3. Partial Sum: $\displaystyle\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$ for $|r|<1$

a. Taking $y = e^{i(x-t)}$, $\displaystyle\sum_{n=-N}^{N} y^n = \sum_{n=0}^{N} y^n + \sum_{n=0}^{N} (y^{-1})^n - 1$ ← since $n=0$ term is double-counted!

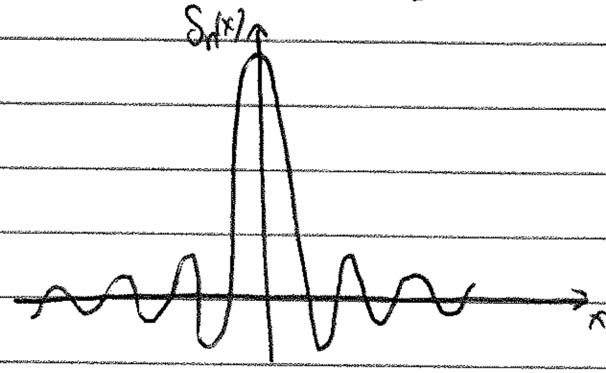
b. Thus $= \dfrac{1-y^{N+1}}{1-y} + \dfrac{1-y^{-N-1}}{1-y^{-1}} - 1 = \dfrac{\cancel{y^{-\frac{1}{2}}} - y^{N+\frac{1}{2}}}{y^{-\frac{1}{2}}-y^{\frac{1}{2}}} + \dfrac{\cancel{-y^{\frac{1}{2}}} + y^{-N-\frac{1}{2}}}{-y^{\frac{1}{2}}+y^{-\frac{1}{2}}} - \dfrac{\cancel{y^{-\frac{1}{2}}} - \cancel{y^{\frac{1}{2}}}}{y^{-\frac{1}{2}}-y^{\frac{1}{2}}}$

$$= \frac{y^{-(N+\frac{1}{2})} - y^{N+\frac{1}{2}}}{y^{-\frac{1}{2}} - y^{\frac{1}{2}}}$$

4. Plugging back in $y = e^{i(x-t)}$ and simplifying, we obtain

$$\boxed{f_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\frac{\sin\left[(N+\frac{1}{2})(x-t)\right]}{\sin\left[\frac{1}{2}(x-t)\right]}\,dt}$$

5. Note: $\delta_n(x) = \dfrac{1}{2\pi}\,\dfrac{\sin\left[(n+\frac{1}{2})x\right]}{\sin\frac{1}{2}x}$ is the Dirichlet kernel of the delta function (1.156), $\lim_{n\to\infty}\delta_n(x) = \delta(x)$.



C. Application to Square Wave

1. For a periodic square wave $f(x) = \begin{cases} \frac{h}{2} & 0 < x < \pi \\ -\frac{h}{2} & -\pi < x < 0, \end{cases}$

the Fourier series is $f(x) = \dfrac{2h}{\pi}\displaystyle\sum_{n=0}^{\infty} \dfrac{\sin(2n+1)x}{(2n+1)}$

2. Applying the equation for $f_N(x)$ (I.B.4.) above to $f(x)$ here,

$$f_N(x) = \frac{h}{4\pi}\int_0^{\pi} \frac{\sin\left[(N+\frac{1}{2})(x-t)\right]}{\sin\left[\frac{1}{2}(x-t)\right]}\,dt - \frac{h}{4\pi}\int_{-\pi}^{0} \frac{\sin\left[(N+\frac{1}{2})(x-t)\right]}{\sin\left[\frac{1}{2}(x-t)\right]}\,dt$$

3. Substituting $x-t=s$ and $x-t=-s$ and rearranging (see text) yields

$$f_N(x) = \frac{h}{4\pi}\int_{-x}^{x} \frac{\sin\left[(N+\frac{1}{2})s\right]}{\sin(\frac{1}{2}s)}\,ds - \frac{h}{4\pi}\int_{-\pi-x}^{\pi+x} \frac{\sin\left[(N+\frac{1}{2})s\right]}{\sin(\frac{1}{2}s)}\,ds$$

I. C. (Continued)

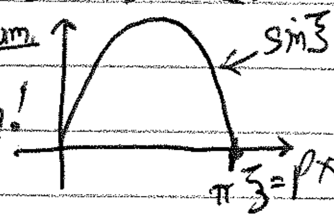
4. For small x, Right-hand integral has denominator $\lim_{x\to 0} \sin\left(\frac{-\pi \pm x}{2}\right) = -1$, but left-hand side has denominator $\lim_{x\to 0} \sin\frac{x}{2} = 0$.
   a. Thus left-hand integral dominates!

5. Define $p = N + \frac{1}{2}$ and $\xi = ps$, use even integrand to obtain

$$\boxed{f_N(x) \approx \frac{h}{2\pi}\int_0^{px} \frac{\sin\xi}{\sin\left(\frac{\xi}{2p}\right)}\frac{d\xi}{p}}$$

← Use this approximation to find value of overshoot for square wave.

6. Properties:
   a. For $x=0$, $f_N(x) = 0 \rightarrow$ average of $\frac{h}{2}$ and $-\frac{h}{2}$.
   b. For $x>0$, integral increases until $\rightarrow$ Num. $px = \pi$. $\Rightarrow$ $x = \frac{\pi}{p}$ is position of maximum!



7. For a given $N \gg 1$, we have $p \gg 1$,
   So we can approximate $\sin\left(\frac{\xi}{2p}\right) \simeq \frac{\xi}{2p}$ since $\frac{\xi}{2p} \ll 1$, to get

$$f_N\left(\frac{\pi}{p}\right) \approx \frac{h}{\pi}\int_0^{\pi} \frac{\sin\xi}{\xi}\,d\xi$$

maximum position!

8. NOTE: a. $\int_0^{\infty} \frac{\sin\xi}{\xi}\,d\xi = \frac{\pi}{2}$

Definite integral!

   b. $\int_0^{\infty} \frac{\sin\xi}{\xi}\,d\xi = \int_0^{\pi} \frac{\sin\xi}{\xi}\,d\xi + \underbrace{\int_{\pi}^{\infty} \frac{\sin\xi}{\xi}\,d\xi}_{= -\mathrm{si}(\pi)} = \frac{\pi}{2}$

← Sine Integral Function.

9. Thus, maximum of overshoot (at $x = \frac{\pi}{p}$) is:

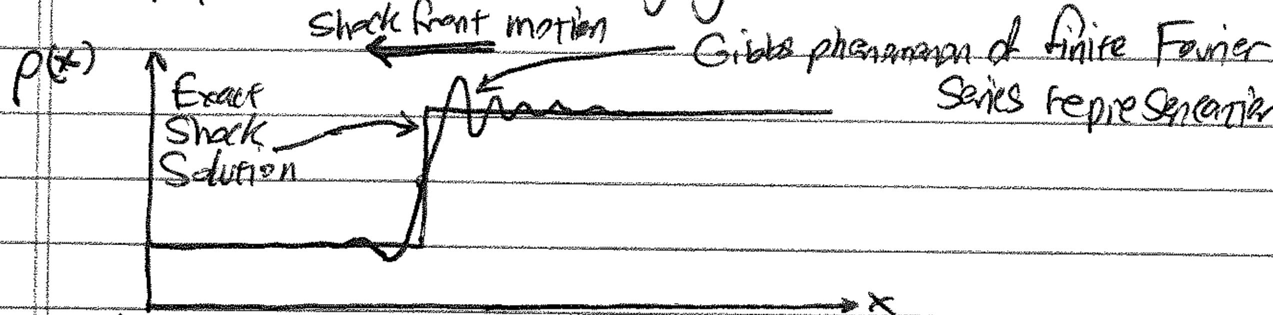
$$f_N\left(\frac{\pi}{p}\right) = \frac{h}{2}\left[\frac{2}{\pi}\left\{\frac{\pi}{2} + \mathrm{si}(\pi)\right\}\right] = \frac{h}{2}\left[1 + \frac{2}{\pi}\mathrm{si}(\pi)\right] = \frac{h}{2}\left[1.1789797\ldots\right]$$

$\Rightarrow$ **Approximately 18% overshoot, independent of N!**

I. (Continued)

D. Consequences of Gibbs Phenomenon

1. For the numerical solution of systems that generate discontinuities, such as calculation of shock dynamics, a finite Fourier Series representation with lead to unphysical overshoot and ringing.



2. Any numerical represenation is always discrete (finite N), so this is unavoidable with Fourier methods.

3. Often codes use special "monotonic" approaches to capture shocks accurately.
   a. TVD - (Total Variation Diminishing) methods
   b. PPM (Piecewise Parabolic Method) method
   c. Lower order solver (1st order) in shock vicinity, higher-order away from shock.

II. Integral Transforms

A. General Properties

1. Define: Integral Transform $\quad g(x) = \int_a^b f(t) K(x,t)\, dt$

   a. Kernel: $K(x,t)$
   b. Limits: a, b

   $\}$ Same for all function pairs $f(t)$ & $g(x)$.

2a. Can be symbolically written as a linear operator, $\quad g(x) = \mathcal{L} f(t)$

II. A.2. (Continued)

b. Linearity: i. $\mathcal{L}[f_1(t)+f_2(t)] = \mathcal{L}f_1(t) + \mathcal{L}f_2(t) = g_1(x) + g_2(x)$

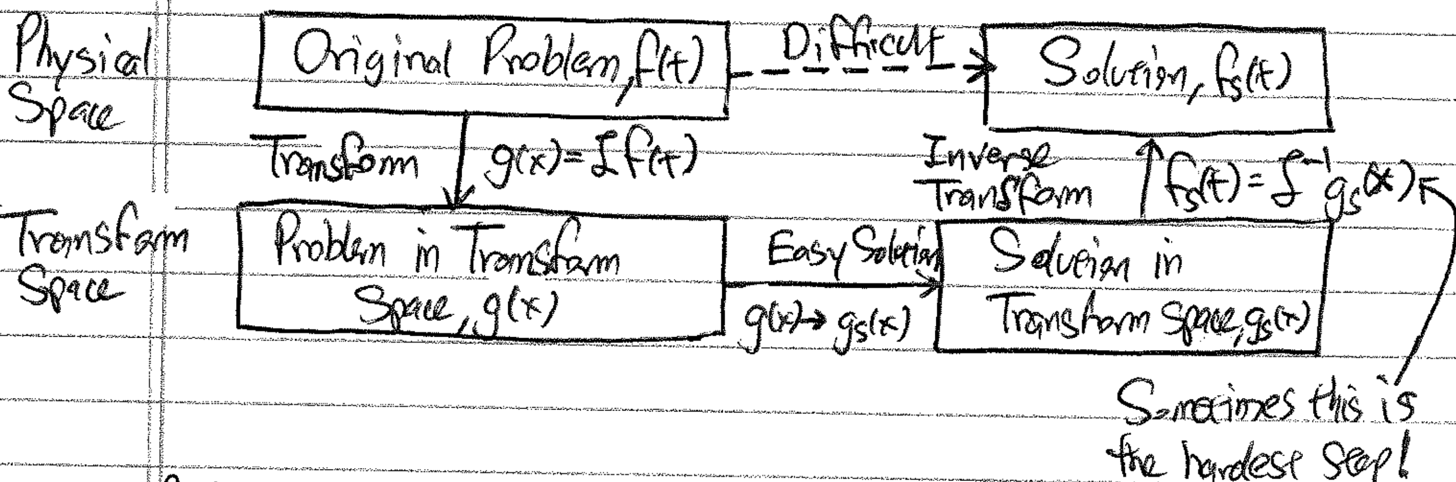
ii. $\mathcal{L}[cf(t)] = c\mathcal{L}f(t) = cg(x)$

3. Inverse Transform: $f(t) = \mathcal{L}^{-1} g(x)$

a. Formula for $\mathcal{L}^{-1}$ depends on specific properties of $K(x,t)$.

4. Utility of Integral Transforms

a. A problem that is difficult to solve in ordinary (physical) space may be much more easy to solve in Transform space.



B. Important Integral Transforms

1. Fourier Transform: $\boxed{g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt}$

a. Beware of variations in definitions of transform & inverse.

2. Laplace Transform: $\boxed{F(s) = \int_0^{\infty} e^{-ts} f(t)\, dt}$

a. Useful for incorporating initial conditions in the solution, but inverse is hard.

3. Hankel Transform: $g(\alpha) = \int_0^{\infty} f(t)\, t\, J_n(\alpha t)\, dt$

III. Fourier Transform

A. Definition

1. Def: Fourier Transform $\boxed{g(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt}$

2. For $f(t)$ of specific parity in t:

a. Even: $g_c(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\cos\omega t\,dt$ b. Odd: $g_s(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin\omega t\,dt$.

Fourier Cosine Transform — Fourier Sine Transform

3. Example: Fourier transform of $f(t)=e^{-\alpha|t|}$ where $\alpha>0$.

a. $g(\omega)=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{+\alpha t}e^{i\omega t}\,dt+\int_0^{\infty} e^{-\alpha t}e^{i\omega t}\,dt\right]$ ← $\boxed{\text{split integral to handle absolute value}}$

$$=\frac{1}{\sqrt{2\pi}}\left\{\left[\frac{e^{(\alpha+i\omega)t}}{\alpha+i\omega}\right]_{-\infty}^{0}+\left[\frac{e^{(-\alpha+i\omega)t}}{-\alpha+i\omega}\right]_0^{\infty}\right\}=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{\alpha+i\omega}+\frac{-1}{-\alpha+i\omega}\right]$$

b. $\boxed{g(\omega)=\sqrt{\frac{1}{2\pi}}\,\frac{2\alpha}{\alpha^2+\omega^2}}$

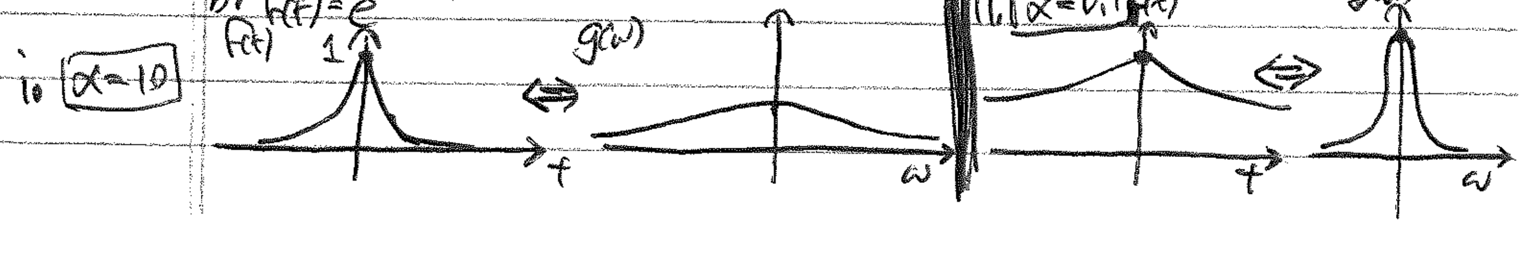
c. NOTE: Since $f(t)$ is even, Fourier Transform is real!

4. Example: $f(t)=\delta(t)$

a. $g(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(t)e^{i\omega t}\,dt=\frac{e^{i\omega(0)}}{\sqrt{2\pi}}=\boxed{\frac{1}{\sqrt{2\pi}}=g(\omega)}$

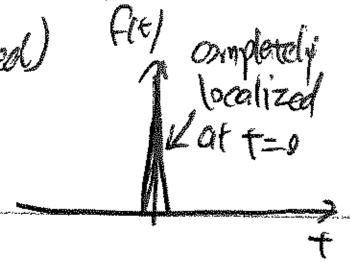
5. Localization in t & ω: a. A localized function in t is spread out in ω, and vice-versa.

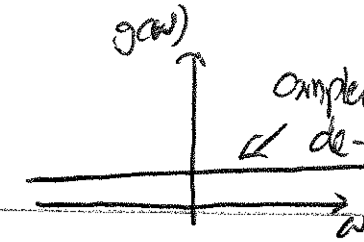
b. $f(t)=e^{-\alpha|t|}$



III. A.5. (Continued)

c. $f(t) = \delta(t)$





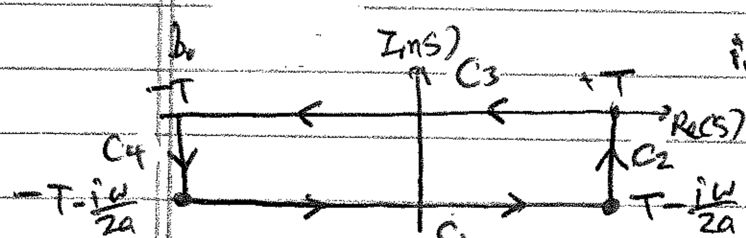
6. <u>Ex:</u> Fourier Transform of a Gaussian: $f(t) = e^{-at^2}$, $a>0$.

a. $$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-at^2} e^{i\omega t}\, dt$$

i. Complete the square: $-at^2 + i\omega t = -at^2 + i\omega t + \frac{\omega^2}{4a} - \frac{\omega^2}{4a} = -a\left[t - \frac{i\omega}{2a}\right]^2 - \frac{\omega^2}{4a}$

ii. $$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{4a}} \lim_{T\to\infty} \int_{-T-i\omega/2a}^{T-i\omega/2a} e^{-as^2}\, ds \quad \text{where } s = t - \frac{i\omega}{2a}$$

b.



i. Consider $\oint_C e^{-as^2}\, ds = 0$ ← No poles enclosed!

$$= \int_{C_1} f(s)\,ds + \cancel{\int_{C_2} f(s)\,ds}^{\,0} + \int_{C_3} f(s)\,ds + \cancel{\int_{C_4} f(s)\,ds}^{\,0}$$

$\lim_{T\to\infty}$ yields nothing for C_2, C_4.

ii. Thus $$\int_{C_1} f(s)\,ds = -\int_{C_3} f(s)\,ds = \underbrace{-\int_{\infty}^{-\infty} e^{-as^2}\,ds}_{\text{Definite integral on real axis!}} = \sqrt{\frac{\pi}{a}}$$

b. Thus, $$\boxed{g(\omega) = \frac{1}{\sqrt{2a}} e^{-\left(\frac{\omega^2}{4a}\right)}}$$ ← <u>Gaussian in ω-space!</u>

c. Full width, half-maximum (FWHM) at exponent $= -\ln 2$, so

i. FWHM of $f(t)$ is $t = (\ln 2/a)^{\frac{1}{2}}$

ii. FWHM of $g(\omega)$ is $\omega = 2(a \ln 2)^{\frac{1}{2}}$

← inverse behavior of width with respect to a!

III.

B. The Fourier Integral and Inverse Fourier Transforms

1. Delta function: $$\boxed{\delta(t) = \lim_{n\to\infty} \delta_n(t) = \lim_{n\to\infty} \frac{1}{2\pi}\int_{-n}^{n} e^{i\omega t}\, d\omega}$$

2. Using $f(x) = \int_{-\infty}^{\infty} f(t)\,\delta(t-x)\, dt$, substituting for $\delta(t)$ above, and interchanging the order of integration yields

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}\, d\omega \underbrace{\int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt}_{=\sqrt{2\pi}\; g(\omega)}}$$

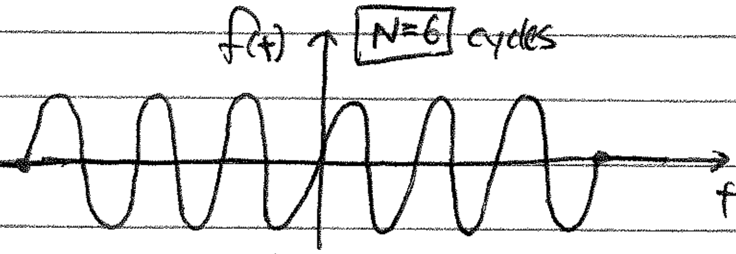
Fourier Integral
[Integral representation of $f(x)$]

3. Def: Inverse Fourier Transform

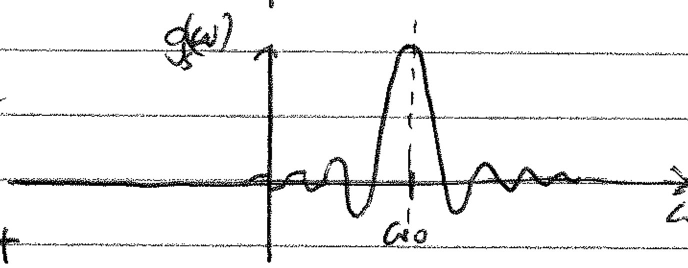
$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)\, e^{-i\omega x}\, d\omega} \iff g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx$$

Fourier Transform

4. Ex: Finite Wave Train



a. $f(t) = \begin{cases} \sin\omega_0 t & |t| < \frac{N\pi}{\omega_0} \\ 0 & |t| > \frac{N\pi}{\omega_0} \end{cases}$



b. Since $f(t)$ is odd, we may use a Fourier Sine Transform:

$$g_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\frac{N\pi}{\omega_0}} \sin(\omega_0 t)\sin(\omega t)\, dt$$

c. Using $\sin A \sin B = \frac{1}{2}\{\cos(A-B) + \cos(A+B)\}$, we may integrate to get

$$g_s(\omega) = \sqrt{\frac{2}{\pi}}\left\{\frac{\sin\left[(\omega_0-\omega)\frac{N\pi}{\omega_0}\right]}{2(\omega_0-\omega)} - \frac{\sin\left[(\omega_0+\omega)\frac{N\pi}{\omega_0}\right]}{2(\omega_0+\omega)}\right\}$$

d. For long pulse (large N), frequency spread is small.
For short pulse (small N), frequency spread is large.
} Related to Heisenberg Uncertainty Principle!