Lecture #16: Fourier Transforms: Properties and Convolutions

I. 3D Fourier Transforms

A. Forward and Inverse

1. For a position $\underset{\sim}{r}$ and wavevector $\underset{\sim}{k}$,

$$\boxed{\begin{aligned} g(\underset{\sim}{k}) &= \frac{1}{(2\pi)^{3/2}} \int f(\underset{\sim}{r})\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}\, d^3\underset{\sim}{r} \\ f(\underset{\sim}{r}) &= \frac{1}{(2\pi)^{3/2}} \int g(\underset{\sim}{k})\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}}\, d^3\underset{\sim}{k} \end{aligned}}$$

2. Plane-wave decomposition:
For a special waveform $f(\underset{\sim}{r})$, the Fourier transform decomposes it into a continuum of plane waves $e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}}$, where each component with wavevector $\underset{\sim}{k}$ has (complex) amplitude $g(\underset{\sim}{k})$.

B. Example 3D Transforms

1. Example: Yukawa Potential $f(\underset{\sim}{r}) = \dfrac{e^{-\alpha r}}{r}$ ← r in spherical coordinates (r,θ,ϕ)

a.
$[\;]^T$ denotes Fourier Transform →
$$\left[\frac{e^{-\alpha r}}{r}\right]^T = \frac{1}{(2\pi)^{3/2}} \int \frac{e^{-\alpha r}}{r} e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}\, d^3\underset{\sim}{r}$$

b. Express volume integral in spherical coordinates,
$$\int d^3\underset{\sim}{r} = \int_0^\infty r^2 dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

c. i. Def: Angular Notation: $\Omega \equiv (\theta,\phi)$
such that $\displaystyle\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = \int d\Omega$ ← 2D integral over (θ,ϕ)

ii. If associated with direction of a particular vector, $\underset{\sim}{r} \Rightarrow \Omega_r$

d. Spherical Wave Expansion of $e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}$

$$e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l j_l(kr) Y_l^m(\Omega_k) Y_l^{*m}(\Omega_r) \qquad \text{Eq (16.61)}$$

e. Using this, we write in spherical coordinates

$$\left[\frac{e^{-\alpha r}}{r}\right]^T = \frac{4\pi}{(2\pi)^{3/2}} \sum_{l,m} i^l \int_0^\infty r e^{-\alpha r} j_l(kr)\,dr\; Y_l^m(\Omega_k) \int d\Omega_r Y_l^{*m}(\Omega_r)$$

Not integrating over $\Omega_k \rightarrow$ hold constant!

f. Simplify $\int d\Omega_r Y_l^{m*}(\Omega_r)$ using orthogonality

i. Since $Y_0^0 = \frac{1}{\sqrt{4\pi}}$, we may write $\sqrt{4\pi}\int d\Omega_r Y_0^0(\Omega_r) Y_l^{m*}(\Omega_r) = \delta_{l0}\delta_{m0}\sqrt{4\pi}$

by orthogonality of $Y_l^m(\Omega)$.

g. Thus, we may eliminate $\sum_{l,m}$ using $\delta_{l0}\delta_{m0}$ to obtain

$$\left[\frac{e^{-\alpha r}}{r}\right]^T = \frac{4\pi}{(2\pi)^{3/2}} \int_0^\infty r e^{-\alpha r} j_0(kr)\,dr$$

h. Since $j_0(x) = \frac{\sin x}{x}$, we get $= \frac{4\pi}{(2\pi)^{3/2}} \int_0^\infty \cancel{r} e^{-\alpha r} \frac{\sin(kr)}{k\cancel{r}}\,dr$

i. Using $\int_0^\infty e^{-ax}\sin bx = \frac{b}{a^2+b^2}$, we obtain a final result

$$\boxed{\left[\frac{e^{-\alpha r}}{r}\right]^T = \frac{1}{(2\pi)^{3/2}} \frac{4\pi}{k^2+\alpha^2}}$$

2. Ex: Coulomb Potential $f(\underset{\sim}{r}) = \frac{1}{r}$

a. Taking $\alpha = 0$ for Yukawa potential result,

$$\boxed{\left[\frac{1}{r}\right]^T = \frac{1}{(2\pi)^{3/2}} \frac{4\pi}{k^2}}$$

I. B. (Continued)

3. Other Useful Examples

a. $$\left[\frac{1}{r^2}\right]^T = \sqrt{\frac{\pi}{2}}\,\frac{1}{k}$$

b. Gaussian: $$\left[e^{-ar^2}\right]^T = \frac{1}{(2a)^{\frac{3}{2}}}\, e^{-\frac{k^2}{4a}}$$

II. Properties of Fourier Transforms

A. Useful Properties

1. Consider a Fourier transform pair: $f(\underset{\sim}{r})$, $g(\underset{\sim}{k})$

a. Translation $\left[f(\underset{\sim}{r}-\underset{\sim}{R})\right]^T = e^{i\underset{\sim}{k}\cdot\underset{\sim}{R}}\, g(\underset{\sim}{k})$

b. Scale $\left[f(\alpha\underset{\sim}{r})\right]^T = \frac{1}{\alpha^3}\, g\left(\frac{\underset{\sim}{k}}{\alpha}\right)$

c. Sign Change $\left[f(-\underset{\sim}{r})\right]^T = g(-\underset{\sim}{k})$

d. Complex Conjugate $\left[f^*(-\underset{\sim}{r})\right]^T = g^*(\underset{\sim}{k})$

e. Gradient $\left[\nabla f(\underset{\sim}{r})\right]^T = -i\underset{\sim}{k}\, g(\underset{\sim}{k})$

f. Laplacian $\left[\nabla^2 f(\underset{\sim}{r})\right]^T = -k^2\, g(\underset{\sim}{k})$

2. 1-D: Fourier Transform Pair: $f(t)$, $g(\omega)$

a. Derivative: $\left[\frac{\partial}{\partial t} f(t)\right]^T = -i\omega\, g(\omega)$

b. n-th Derivative $\left[\frac{\partial^n}{\partial t^n} f(t)\right]^T = (-i\omega)^n g(\omega)$

3. Showing translation: $f(\underset{\sim}{r}-\underset{\sim}{R})$ where $\underset{\sim}{R}$ is constant

a. Start with definition of Fourier transform $\left[f(\underset{\sim}{r}-\underset{\sim}{R})\right]^T = \frac{1}{(2\pi)^{\frac{3}{2}}}\int f(\underset{\sim}{r}-\underset{\sim}{R})e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}d^3\underset{\sim}{r}$

b. Subsitute $\underset{\sim}{r}=\underset{\sim}{r}'+\underset{\sim}{R}$. Since $\underset{\sim}{R}=\text{const}$, $d^3\underset{\sim}{r}=d^3\underset{\sim}{r}'$, so

$$= \frac{1}{(2\pi)^{\frac{3}{2}}}\int f(\underset{\sim}{r}')e^{i\underset{\sim}{k}\cdot(\underset{\sim}{r}'+\underset{\sim}{R})}d^3\underset{\sim}{r}' = \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{R}}}{(2\pi)^{\frac{3}{2}}}\underbrace{\int f(\underset{\sim}{r}')e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}'}d^3\underset{\sim}{r}'}_{=(2\pi)^{\frac{3}{2}}g(\underset{\sim}{k})}$$

$$= e^{i\underset{\sim}{k}\cdot\underset{\sim}{R}}\, g(\underset{\sim}{k}) \checkmark$$

4. For derivative, take definition of inverse transform

$$\nabla f(\underset{\sim}{r}) = \nabla\left\{\frac{1}{(2\pi)^{\frac{3}{2}}}\int g(\underset{\sim}{k})e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}}d^3\underset{\sim}{k}\right\} = \frac{1}{(2\pi)^{\frac{3}{2}}}\int g(\underset{\sim}{k})\underbrace{\left[\nabla_{\underset{\sim}{r}}\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}}\right]}_{=-i\underset{\sim}{k}e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}}}d^3\underset{\sim}{k}$$

$$= -i\underset{\sim}{k}\,\frac{1}{(2\pi)^{\frac{3}{2}}}\int g(\underset{\sim}{k})e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}}d^3\underset{\sim}{k} = -i\underset{\sim}{k}\,g(\underset{\sim}{k}) \checkmark$$

Use Fourier Transforms to Convert PDEs to ODEs!

B. Example: Waves on an Infinite String

1. Wave Eq: $\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$ PDE

2. Initial Conditions: $y(x,0)=f(x)$ $\quad \left.\frac{\partial y}{\partial t}\right|_{t=0}=0$

3. Assume localized disturbance: $\lim_{x\to\pm\infty} f(x)=0$!

4. Fourier transform equation: $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dx\, e^{ikx}$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2 y}{\partial x^2}e^{ikx}dx = \frac{1}{v^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2 y}{\partial t^2}e^{ikx}dx$$

II. B. (Continued)

5. Take $Y(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(x,t) e^{ikx} dx$ (Fourier Transform)

$\Rightarrow$ Fourier transform pair: $y(x,t) \Leftrightarrow Y(k,t)$

6. a. Derivative property (II.A.2.b.) yields

$$\left[\frac{\partial^2 y}{\partial x^2}\right]^T = (-ik)^2 Y(k,t) = -k^2 Y(k,t)$$

b. RHS: $\frac{1}{\sqrt{2\pi}} \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} y e^{ikx} dx = \frac{1}{v^2} \frac{\partial^2 Y(k,t)}{\partial t^2}$

7. Thus, our transformed equation is

$$\boxed{-k^2 Y = \frac{1}{v^2} \frac{\partial^2 Y}{\partial t^2}}$$ PDE has been converted to ODE!

Solution: $Y(k,t) = A e^{ikvt} + B e^{-ikvt}$

8. Apply Initial Conditions:

a. We need to transform ICs to Fourier Space:

$$Y(k,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{f(x)}_{y(x,0)} e^{ikx} dx \equiv F(k), \quad \left.\frac{dY(k,0)}{dt}\right|_{t=0} = 0!$$

b. Thus, we know Fourier initial conditions $Y(k,0) = F(k)$, $\left.\frac{dY}{dt}\right|_{t=0} = 0$

c. $Y(k,0)$: $\Rightarrow$ $F(k) = A + B$

d. $\frac{dY}{dt} = ikvAe^{ikvt} - ikvBe^{-ikvt} \Rightarrow \left.\frac{dY}{dt}\right|_{t=0} = ikv(A-B) = 0 \Rightarrow A = B.$

e. Thus, $F(k) = A + B = A + (A) = 2A \Rightarrow A = B = \frac{F(k)}{2}$

9. Fourier Solution: $\boxed{Y(k,t) = F(k) \frac{e^{ikvt} + e^{-ikvt}}{2} = F(k)\cos(kvt)}$

10. Inverse Fourier Transform Solution (k back to x)

$$y(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[F(k)\,\frac{e^{ikvt}+e^{-ikvt}}{2}\right]e^{-ikx}\,dk$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{F(k)}{2}\left[e^{-ik(x-vt)}+e^{-ik(x+vt)}\right]dk$$

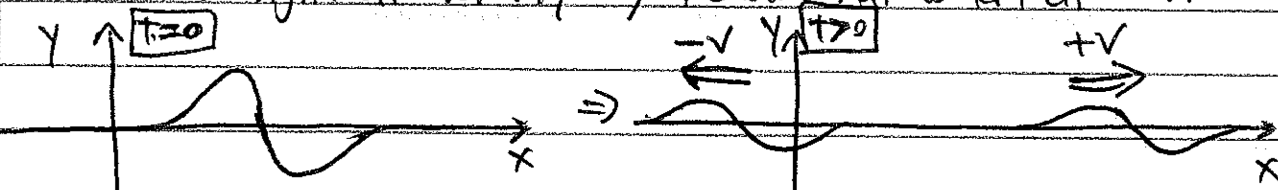
11. For variable $y = x - vt$, the first term is

$$\frac{1}{2}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(k)\,e^{-iky}\,dk\right\} = \frac{1}{2}f(y) = \frac{1}{2}f(x-vt)$$

12. Thus, the final general solution is:

$$\boxed{y(x,t) = \frac{1}{2}\left[f(x-vt) + f(x+vt)\right]}$$

13. Physical Interpretation: Half amplitude of initial wave form $y(x,0)=f(x)$ moves to right at velocity v, the other half to left at $-v$.



14. NOTE: We did not ever need to know form $f(x)$ to solve
⇒ consequence of properties of wave equation!

C. Example: Coulomb Green's Function

1. $\nabla_r^2 G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$

2. Fourier Transform both sides

a. $\left[\nabla_r^2 G(\mathbf{r},\mathbf{r}')\right]^T = -k^2 g(\mathbf{k},\mathbf{r}')$ where $g(\mathbf{k},\mathbf{r}')$ is Fourier Transform of $G(\mathbf{r},\mathbf{r}')$

NOTE: $\mathbf{r}'$ is held constant → unaffected!

b. $[\delta(\underset{\sim}{r}-\underset{\sim}{r}')]^T = e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}'}[\delta(\underset{\sim}{r})]^T$ by translation property.

i. NOTE: $[\delta(\underset{\sim}{r})]^T = \frac{1}{(2\pi)^3}\int \delta(\underset{\sim}{r})\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} d^3\underset{\sim}{k} = \frac{1}{(2\pi)^3}$

3. Thus, we obtain the transformed equation $-k^2 g(\underset{\sim}{k},\underset{\sim}{r}') = \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}'}}{(2\pi)^{3/2}}$

Solution: $\boxed{g(\underset{\sim}{k},\underset{\sim}{r}') = -\frac{1}{(2\pi)^{3/2}}\frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}'}}{k^2}}$

4. Inverse Fourier Transform:

$$G(\underset{\sim}{r},\underset{\sim}{r}') = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\left[-\frac{1}{(2\pi)^{3/2}}\frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}'}}{k^2}\right]e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}} d^3\underset{\sim}{k} = \frac{-1}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{e^{-i\underset{\sim}{k}\cdot(\underset{\sim}{r}-\underset{\sim}{r}')}}{k^2} d^3\underset{\sim}{k}$$

a. Let $\underset{\sim}{y} = \underset{\sim}{r}-\underset{\sim}{r}'$ $\quad \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{y}}}{k^2} d^3\underset{\sim}{k} \leftarrow$ Inverse Fourier Transform of $\frac{1}{k^2}$

b. Recall: $f(\underset{\sim}{r}) = \frac{1}{r} \iff F(\underset{\sim}{k}) = \frac{1}{(2\pi)^{3/2}}\frac{4\pi}{k^2}$ (Transform pair)

c. Thus, we obtain $\frac{(2\pi)^{3/2}}{4\pi}\left\{\frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\left[\frac{1}{(2\pi)^{3/2}}\frac{4\pi}{k^2}\right]e^{-i\underset{\sim}{k}\cdot\underset{\sim}{y}} d^3\underset{\sim}{k}\right\} = \frac{(2\pi)^{3/2}}{4\pi}\frac{1}{y}$

d. Since $y = |\underset{\sim}{y}| = |\underset{\sim}{r}-\underset{\sim}{r}'|$, we put it all together to obtain

$$\boxed{G(\underset{\sim}{r},\underset{\sim}{r}') = -\frac{1}{4\pi}\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}}$$

5. General Usefulness of Fourier Transforms

a. Can convert a PDE into an ODE

b. Can convert a difficult problem into one easy to solve. Then, transform back to original space.

III. Fourier Convolution Theorem

A. Definition

1. Def: Convolution of $f(x)$ & $g(x)$, $f*g$

a. 1D: $$\boxed{(f*g)(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y)\, f(x-y)\, dy}$$

function of x

b. 3-D: $$(f*g)(\underset{\sim}{r}) = \frac{1}{(2\pi)^{3/2}} \int g(\underset{\sim}{r}')\, f(\underset{\sim}{r}-\underset{\sim}{r}')\, d^3\underset{\sim}{r}'$$

2. Fourier Transform of convolution: $(f*g)(x) \Leftrightarrow (f*g)^T(k)$

a. $$\left[(f*g)(x)\right]^T = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y)\, f(x-y)\, dy\right] e^{+ikx}$$

b. Write $e^{ikx} = e^{iky + ik(x-y)} = e^{iky} e^{ik(x-y)}$, so

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} dy\, g(y)\, e^{iky} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x-y)\, e^{ik(x-y)} \right] \right\}$$

c. Taking $z = x-y$, we find $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, f(z)\, e^{ikz} = F(k)$

$y = \text{const} \rightarrow dz = dx$

No longer a function of y or x!

d. Thus
$$= G(k)\, F(k)$$

e. Summarizing $$\boxed{\left[(f*g)(x)\right]^T = G(k)\, F(k)}$$ Convolution Theorem

where $g(x) \Leftrightarrow G(k)$ and $f(x) \Leftrightarrow F(k)$ are Fourier transform pairs.

III. A. (Continued)

3. From Inverse Fourier Transform $\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y) f(x-y)\,dy}_{(f*g)(x)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{G(k)F(k)}_{[(f*g)(x)]^T} e^{-ikx}\,dk$

we obtain $$\boxed{\int_{-\infty}^{\infty} g(y) f(x-y)\,dy = \int_{-\infty}^{\infty} G(k)F(k)e^{-ikx}\,dk}$$

a. Two functions in original integral had different arguments (evaluate at diff. points) but convolution theorem yields integrand of two functions at <u>same point</u>!

b. The cost is to introduce oscillatory term e^{-ikx}

c. In 3D $\boxed{\int g(\underset{\sim}{x}') f(\underset{\sim}{x}-\underset{\sim}{x}')\,d^3\underset{\sim}{x}' = \int F(\underset{\sim}{k})G(\underset{\sim}{k}) e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}}\,d^3\underset{\sim}{k}}$

B. <u>Parseval Relation</u>

1. Taking Convolution Thm at $x=0$, $\int_{-\infty}^{\infty} h(-y) g(y)\,dy = \int_{-\infty}^{\infty} H(k)G(k)\,dk$

a. If we take $f^*(y) = h(-y)$, then $[f^*(-y)]^T = F^*(k)$, so we obtain,

$$\boxed{\int_{-\infty}^{\infty} f^*(y) g(y)\,dy = \int_{-\infty}^{\infty} F^*(k)G(k)\,dk}$$ Parseval Relation

2. Fourier Transform is <u>Unitary</u>.

a. Writing Symbolically as scalar product, $\langle f|g\rangle = \langle \mathcal{F}f|\mathcal{F}g\rangle$ (Fourier Transform operator)

b. Moving $\mathcal{F}$ operator from left to right, $\langle f|g\rangle = \langle f|\mathcal{F}^\dagger\mathcal{F}g\rangle$

c. Since this is true for all f, g in Hilbert space, $\mathcal{F}^\dagger\mathcal{F} = 1 \Rightarrow \boxed{\mathcal{F}^\dagger = \mathcal{F}^{-1}}$ Fourier Transform is <u>Unitary</u>

3. Using $g=f$, we obtain $\boxed{\langle f|f\rangle = \langle F|F\rangle}$

a. Parseval Relation corresponds to <u>Conservation of Energy</u>.

b. Energy is the same, whether the function f is expressed in physical space $f(x)$ or Fourier space $F(k)$!