Lecture #19: Laplace Transforms: More Properties and Convolution Theorem

I. Laplace Transforms: More Properties

A. Translation:

1. Consider a time signal $f(t)$ with Laplace Transform $F(s)=\mathcal{L}\{f(t)\}$
2. Now, what happens if we multiply by e^{-bs}?

a. $$e^{-bs}F(s)=e^{-bs}\int_0^\infty dt\, e^{-st}f(t)=\int_0^\infty dt\, f(t)\, e^{-s(t+b)}$$

b. Change variable $\tau=t+b \Rightarrow t=\tau-b$

$$e^{-bs}F(s)=\int_b^\infty d\tau\, f(\tau-b)\, e^{-s\tau}$$

c. But, we assume $f(t)=0$ for $t<0$, so

$$\int_b^\infty d\tau\, f(\tau-b)e^{-s\tau}=\int_0^\infty d\tau\, f(\tau-b)e^{-s\tau}.$$

d. Thus $\boxed{e^{-bs}F(s)=\mathcal{L}\{f(\tau-b)\}}$ Translation of argument

3. Instead of relying on $f(t)=0$ for $t<0$, insert Heaviside function, $U(\tau-b)=\begin{cases}1 & \tau>b\\ 0 & \tau<b\end{cases}$

$$e^{-bs}F(s)=\int_0^\infty d\tau\, e^{-s\tau}f(\tau-b)\,U(\tau-b) \leftarrow \text{Heaviside Shifting Theorem}$$

4. Example: Electromagnetic Waves

a. For an electromagnetic wave (E_y or $E_z \to E$) propagating in x-direction,

$$\frac{\partial^2 E(x,t)}{\partial x^2}-\frac{1}{v^2}\frac{\partial^2 E(x,t)}{\partial t^2}=0$$

b. Source: At $x=0$, $E_s(0,t)$ is given

c. Initial Conditions: $E(x,0)=0 \qquad \left.\frac{\partial E(x,t)}{\partial t}\right|_{t=0}=0$

a. Assume wave propagates on in +x direction

c. Performing Laplace transform in time:

$$\frac{\partial^2}{\partial x^2}\mathcal{L}\{E(x,t)\} - \frac{s^2}{v^2}\mathcal{L}\{E(x,t)\} + \cancel{\frac{s}{v^2}E(x,0)}^{\,0} + \cancel{\frac{1}{v^2}\frac{\partial E(x,t)}{\partial t}\bigg|_{t=0}}^{\,0} = 0$$

d. $$\frac{\partial^2}{\partial x^2}\mathcal{L}\{E(x,t)\} = \frac{s^2}{v^2}\mathcal{L}\{E(x,t)\}$$

e. General Solution: $$\mathcal{L}\{E(x,t)\} = F_1(s)\, e^{-\left(\frac{s}{v}\right)x} + F_2(s)\, e^{+\left(\frac{s}{v}\right)x}$$

Corresponds to different propagation directions!

f. Let $F_2(s) = 0$, so $\mathcal{L}\{E(x,t)\} = F_1(s)\, e^{-\left(\frac{s}{v}\right)x}$

g. By translation property $\mathcal{L}^{-1}\{F_1(s)\, e^{-\left(\frac{s}{v}\right)x}\} = E(x, t-\frac{x}{v})$ $\quad b = \frac{x}{v}$

i. At $x=0$, $\mathcal{L}^{-1}\{F_1(s)\} = E(0,t) = E_s(0,t)$

ii. So, $F_1(s) = \mathcal{L}\{E_s(0,t)\}$ ← Known source at $x=0$!

iii. Thus $E(x, t-\frac{x}{v}) = E_s(0, t-\frac{x}{v})$

← translated time!

h. Solution:

$$\boxed{E(x,t) = \begin{cases} E(0, t-\frac{x}{v}) & t \geq \frac{x}{v} \\ 0 & t < \frac{x}{v} \end{cases}}$$

← At $x > vt$, signal has not yet reached position x!

Waveform of $E(0,t)$ moving with velocity v at position $x = vt$!

i. If $F_2(s) \neq 0$, will have component moving at velocity $-v$

⇒ inconsistent with specified condition (a) above!

I. (continued)

B. Derivative of a Laplace Transform

1. If $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ is uniformly convergent,

   we may differentiate $F(s)$ with respect to s,

   a. $F'(s) = \frac{\partial}{\partial s}\left[\int_0^\infty e^{-st} f(t)\,dt\right] = \int_0^\infty -t\,e^{-st} f(t)\,dt = \int_0^\infty e^{-st}[-t f(t)]\,dt$

   b. Thus $\boxed{F^{(n)}(s) = \mathcal{L}\{(-t)^n f(t)\}}$

2. Technique can be used to derive new Laplace Transforms.

   a. Consider $f(t) = e^{kt}$, $F(s) = \int_0^\infty e^{-st} e^{kt}\,dt$ (requires $s>k$ for convergence!)

   b. $F(s) = \frac{1}{s-k}$

   c. $F'(s) = \int_0^\infty e^{-st}[-t e^{kt}]\,dt = \frac{-1}{(s-k)^2} \Rightarrow \boxed{\mathcal{L}\{t e^{kt}\} = \frac{1}{(s-k)^2},\ s>k}$

C. Integration of a Laplace Transform

1. Again, assume $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ is uniformly convergent.

2. Now $\int_s^\infty F(x)\,dx = \int_s^\infty dx \int_0^\infty e^{-xt} f(t)\,dt = \int_0^\infty dt \underbrace{\left[\int_s^\infty e^{-xt}\,dx\right]}_{=\left[\frac{-e^{-xt}}{t}\right]_s^\infty = \left[\cancel{\frac{-e^{-\infty t}}{t}}^{\,0} + \frac{e^{-st}}{t}\right]} f(t)$

   a. NOTE: $F(\infty) = 0$ due to factor e^{-st} in integral!

   $= \int_0^\infty e^{-st}\left[\frac{f(t)}{t}\right] dt$

3. Thus $\boxed{\int_s^\infty F(x)\,dx = \mathcal{L}\left\{\frac{f(t)}{t}\right\}}$

4. NOTE: The limit of the integral s (the transform variable) must be chosen large enough that Laplace transform converges.

5. Also, either $\frac{f(t)}{t}$ must be finite at $t=0$, or diverge less strongly than t^{-1}.

II. Laplace Convolution Theorem

A. Definition

1. Consider two functions $f_1(t)$ and $f_2(t)$ with Laplace transforms $F_1(s) = \mathcal{L}\{f_1(t)\}$ and $F_2(s) = \mathcal{L}\{f_2(t)\}$

2. Multiply transforms

a. $F_1(s) F_2(s) = \left[\int_0^\infty e^{-sx} f_1(x) dx\right]\left[\int_0^\infty e^{-sy} f_2(y) dy\right]$

Same transform variable s

$= \int_0^\infty \int_0^\infty e^{-s(x+y)} f_1(x) f_2(y) dx dy$

b. Change variable: $t = x+y \rightarrow x = t-y$ and $dx = dt$
$u = y \qquad\qquad = t-u$

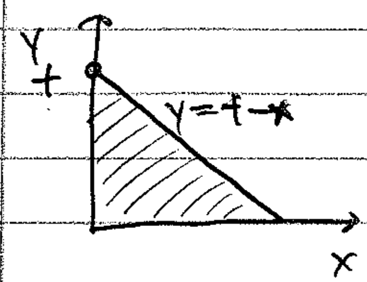
i. $(x,y) \rightarrow (t,u)$

ii. Jacobian $J = \frac{\partial(x,y)}{\partial(t,u)}$ where $dx dy = J dt du$

iii. $J = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1 \Rightarrow dx dy = dt du$

II. A2.b. (continued) Hawes (5)

iv. Limits of integration



a. Choose $\int_0^\infty dt$

b. What range of values is covered by $\int dy$?

i. since $x \geq 0$, and $x = t - y$

$\Rightarrow \int_0^t dy$

c. Thus $\int_0^\infty dx \int_0^\infty dy = \int_0^\infty dt \int_0^t dy$

C. Thus

$$F_1(s) F_2(s) = \int_0^\infty e^{-st} \left[\int_0^t f_1(t-y) f_2(y)\, dy \right] dt$$

$$\boxed{F_1(s) F_2(s) = \mathcal{L}\left\{ \int_0^t f_1(t-y) f_2(y)\, dy \right\} \equiv \mathcal{L}\{ f_1 * f_2 \}}$$

3. Thus, Laplace Convolution $\boxed{f_1 * f_2 \equiv \int_0^t f_1(t-y) f_2(y)\, dy}$

4. Laplace Convolution can be used to:

a. Find new transforms

b. Use as an alternative to partial fraction expansion

B. Ex: Driven, Damped Oscillator

1. Consider a mass m on a spring (k) with damping (b) and driving $f(t)$.

$$m x''(t) + b x'(t) + k x(t) = f(t)$$

driving force applied to system!

2. Choose initial conditions: $x(0) = 0$

$x'(0) = 0$

3. Laplace Transform

$$m s^2 X(s) + bs X(s) + k X(s) = F(s)$$

where $F(s) = \mathcal{L}\{f(t)\}$

Transform of driving.

II. B. (Continued)

4. Solving for $X(s)$: $\boxed{X(s) = \frac{F(s)}{m} \frac{1}{(s+\frac{b}{2m})^2 + \omega_1^2}}$

where $\omega_1^2 = \omega_0^2 - \frac{b^2}{4m^2}$ and $\omega_0^2 = \frac{k}{m}$

5. <u>NOTE:</u> Form of $X(s)$ is a product of two Laplace Transforms!

a. $\frac{F(s)}{m} = \mathcal{L}\left\{\frac{f(t)}{m}\right\}$

b. From (20.158) $\mathcal{L}\{e^{at}\sin kt\} = \frac{k}{(s-a)^2+k^2}$, so

$$\frac{1}{\omega_1} \frac{\omega_1}{(s+\frac{b}{2m})^2+\omega_1^2} = \mathcal{L}\left\{\frac{1}{\omega_1} e^{-\frac{bt}{2m}} \sin \omega_1 t\right\}$$

6. Use Laplace Convolution Theorem to obtain solution in integral form:

$$\boxed{x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{m\omega_1}\int_0^t f(t-y)\, e^{-\frac{by}{2m}} \sin \omega_1 y \, dy}$$

a. <u>NOTE:</u> This form is valid for an <u>aribtrary choice</u> of driving force $f(t)$!

7. Choose particular forcing $f(t)$:

a. $f(t) = P\delta(t)$, where P is momentum imparted by impulse at $t=0$.

b. $x(t) = \frac{1}{m\omega_1}\int_0^t P\,\delta(t-y)\, e^{-\frac{by}{2m}} \sin\omega_1 y\, dy = \boxed{\frac{P}{m\omega_1} e^{-\frac{bt}{2m}} \sin \omega_1 t = x(t)}$