Hawes ①

Lecture #20 Inverse Laplace Transforms

I. Inverse Laplace Transforms

A. Bromwich Integral

1. Given a Laplace transform $F(s)$, we want to determine the inverse

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

where $F(s) = \int_0^\infty e^{-st} f(t)\,dt$

2. NOTE: For Laplace transform to exist, $e^{-st}f(t)$ must not diverge.
   a. Generally this means we must take $s > 0$.
   b. In fact, if $f(t) \propto e^{\alpha t}$, then $e^{-st}f(t) \propto e^{(\alpha - s)t}$ converges if $s > \alpha$

3. Use Fourier transform to derive inverse Laplace transform.
   a. Fourier Integral

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iyt}\,dy \int_{-\infty}^{\infty} f(x) e^{-iyx}\,dx$$

x & y are dummy (integration) variables.

   b. For this to be valid, $F(\omega)$ [Fourier transform of $f(t)$] must satisfy $\lim_{\omega \to \infty} F(\omega) = 0$ so integral of transform converges.

   c. But, we can Laplace transform a divergent function $f(t) \propto e^{\alpha t}$ long as we take $s > \alpha$.

4. To allow Fourier integral to be applied, take

$$f(t) = e^{\beta t} g(t)$$

where $\beta > \alpha$ ensures that $g(t) = e^{-\beta t} f(t) \propto e^{(\alpha - \beta)t}$ converges.

5. Also, extend $g(t)$ to $t < 0$ by defining

$$g(t) = \begin{cases} g(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

since a Fourier transform must be defined over $-\infty < t < \infty$.

I. A. Continued

6. Now, we apply Fourier integral to $g(t)$

a. $g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyt}\, dy \int_{0}^{\infty} g(x) e^{-iyx}\, dx$ ← since $g(x) = 0$ for $x < 0$!

b. Substitute $g(t) = e^{-\beta t} f(t)$

$$f(t) = \frac{e^{\beta t}}{2\pi}\int_{-\infty}^{\infty} e^{iyt}\, dy \int_{0}^{\infty} f(x) e^{-(\beta + iy)x}\, dx$$

7. Define $s = \beta + iy$, so $\int_{0}^{\infty} f(x) e^{-sx}\, dx = F(s)$ ← Usual Laplace Transform.

8. Thus $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\beta+iy)t}\, dy \left[F(s)\right]$

$s = \beta + iy$

$ds = i\,dy \rightarrow dy = \frac{ds}{i}$

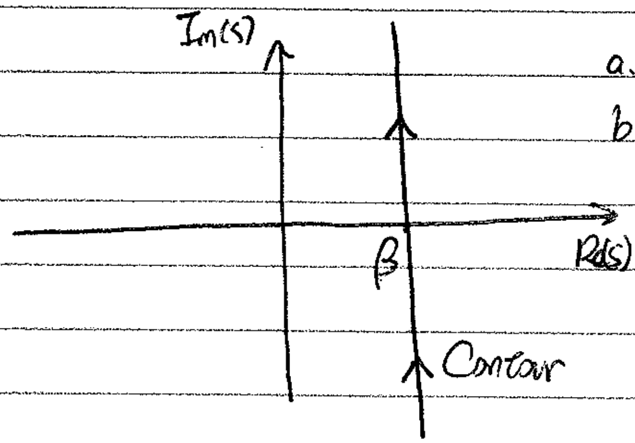
Limits: $s = \beta + i(-\infty) = \beta - i\infty$

$s = \beta + i(\infty) = \beta + i\infty$

Bromwich Integral

$$\boxed{f(t) = \frac{1}{2\pi i}\int_{\beta - i\infty}^{\beta + i\infty} e^{st} F(s)\, ds}$$

9.



a. Here s is complex

b. Converges as long as $Re(s) \geq \beta$

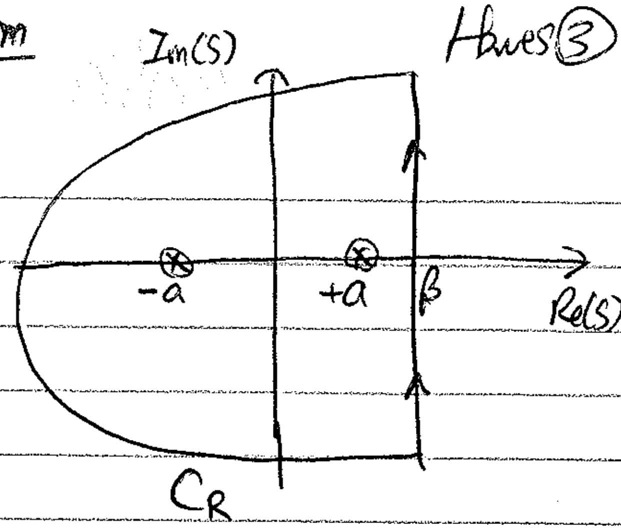
Singularities in $F(s)$ may only exist to the left of contour.

10. We may evaluate the integral by contour integration.

a. In particular, we can close integral with a counter-clockwise contour that closes at $Re(s) = -\infty$.

b. Then we can apply Residue Theorem for poles in $F(s)$ within contour.

I. B. Ex: Inverse Laplace Transform



1. Consider $F(s) = \dfrac{a}{s^2 - a^2}$

2. $$f(t) = \frac{1}{2\pi i}\int_{\beta - i\infty}^{\beta + i\infty} e^{st} F(s)\,ds$$

3a. Poles: $F(s) = \dfrac{a}{(s+a)(s-a)}$

$\boxed{s = \pm a}$

b. Must choose $\beta > a$

4. Consider $$\frac{1}{2\pi i}\oint_C e^{st}F(s)\,ds = \frac{1}{2\pi i}\int_{\beta+i\infty}^{\beta+i\infty} e^{st}F(s)\,ds + \frac{1}{2\pi i}\int_{C_R} e^{st}F(s)\,ds$$

a. NOTE: Since $\lim_{|s|\to\infty} F(s) = 0$, contribution from C_R is zero!

5. By Residue Theorem: $$\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{st}F(s)\,ds = \frac{1}{\cancel{2\pi i}}\cancel{2\pi i}\sum_j \operatorname*{Res}_{s=s_j}\left[\frac{a e^{st}}{(s+a)(s-a)}\right]$$

$$= \frac{\cancel{a}e^{-at}}{-2\cancel{a}} + \frac{\cancel{a}e^{at}}{2\cancel{a}} = \frac{e^{at} - e^{-at}}{2} = \boxed{\sinh at = f(t)}$$

6. Generally:

a. When Laplace transform tables do not help you find an appropriate inverse transform, you can always apply the Bromwich integral

b. Complex contour integration using the Residue Theorem is a powerful approach, showing that poles are associated with terms in the solution

II. Integral Equations

A. Classes of Integral Equations

1a. Differential equations involve an unknown function and its derivatives.

b. Integral equations contain an unknown function within a integral.

2. We shall focus on linear integral equations.

3. Two Classes:

a. Fixed limits of integration: Fredholm equation

b. One limit is a variable: Volterra equation

4. Two Kinds:

a. Unknown function only under integral sign $\Rightarrow$ first kind

b. Unknown function inside & outside integral $\Rightarrow$ second kind

5. Examples:

$\phi(t)$ is unknown function

$K(x,t)$ is kernel

$f(x)$ is a known function (If $f(x)=0 \rightarrow$ homogeneous)

a. Fredholm Eq. of first kind:

$$f(x) = \int_a^b K(x,t)\,\phi(t)\,dt$$

b. Fredholm Eq. of second kind:

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\,\phi(t)\,dt$$

eigenvalue λ

c. Volterra Eq. of first kind:

$$f(x) = \int_a^x K(x,t)\,\phi(t)\,dt$$

d. Volterra Eq. of second kind: $\phi(x) = f(x) + \int_a^x K(x,t)\,\phi(t)\,dt$

6. How do differential & integral equations differ?
a. For DE's, solution must be subject to boundary conditions to specify final answer.
b. For IE's, boundary conditions are built into the equations.
$\Rightarrow$ Integral equation relates solution to value through a region, not just to its derivatives at one point!

7. Many mathematical properties (existence, uniqueness, completeness) are more elegantly handled with equations in integral form!

8. Ex: Momentum Representation of Quantum Mechanics
a. Schrödinger Eq: $-\frac{\hbar^2}{2m}\nabla^2\psi(\underline{r})+V(\underline{r})\psi(\underline{r})=E\,\psi(\underline{r})$
b. Fourier transforming, using Convolution Thm and Hartree Atomic units ($\hbar=m=e=1$),

$$\frac{k^2}{2}\phi(\underline{k})+\frac{1}{(2\pi)^{\frac{3}{2}}}\int\frac{4\pi}{|\underline{k}-\underline{k}'|}\phi(\underline{k}')\,d^3\underline{k}'=E\,\phi(\underline{k})$$

Fredholm eigenvalue Eq. of Second Kind

$\phi(\underline{k})$ unknown function
$K(\underline{k},\underline{k}')=\frac{4\pi}{|\underline{k}-\underline{k}'|}$
E is the eigenvalue

9. Ex: Kinetic Plasma Physics
a. Vlasov Equation $\frac{\partial f_s(x,v)}{\partial t}+v\frac{\partial f_s(x,v)}{\partial x}-\frac{q}{m}\frac{\partial\phi(x)}{\partial x}\frac{\partial f_s(x,v)}{\partial v}=0$

b. Poisson Equation: $-\frac{\partial^2\phi}{\partial x^2}=4\pi\sum\int_{-\infty}^{\infty}dv\,q_s f_s(x,v)$

Integro-differential equations

II. (Continued)

B. Transforming a Differential Equation to an Integral Equation

1. a. $y'' + A(x)y' + B(x)y = g(x)$ Linear 2nd order ODE

   b. Initial conditions: $y(a) = y_0$ $\quad y'(a) = y_0'$ ← $\boxed{\text{Cauchy BC's at } x=a}$

2. Integrate $\int_a^x dt$

   a. $$\int_a^x y''(t)\,dt = \left[y'(t)\right]_a^x = y'(x) - y'(a) = y'(x) - y_0'$$

   b. $$\int_a^x A(t)y'(t)\,dt = \left[A(t)y(t)\right]_a^x - \int_a^x A'(t)y(t)\,dt = A(x)y(x) - A(a)y_0 - \int_a^x A'y\,dt$$

   Integrate by parts $\quad u = A(t) \quad dv = y'(t)dt$, $\quad du = A'(t)dt \quad v = y$

   c. Putting it together

$$y'(x) = -A(x)y(x) - \int_a^x \left[B(t) - A'(t)\right] y(t)\,dt + \int_a^x g(t)\,dt + A(a)y_0 + y_0'$$

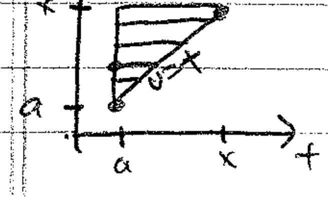
3. Integrating once more $\int_a^x dt$, New dummy integration variable u

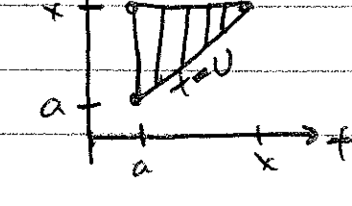
$$y(x) - y_0 = -\int_a^x A(t)y(t)\,dt - \int_a^x du \int_a^u \left[B(t) - A'(t)\right] y(t)\,dt$$
$$+ \int_a^x du \int_a^u g(t)\,dt + \left[A(a)y_0 + y_0'\right] \underbrace{\int_a^x dt}_{=(x-a)}$$

Since original limit was a variable,
It becomes new dummy integration variable u.

4. Eliminate double integrals:

$$\int_a^x du \int_a^u f(t)\,dt = \int_a^x dt\, f(t) \int_t^x du = \int_a^x dt\, f(t)\left[x - t\right]$$
$$= \int_a^x (x-t) f(t)\,dt$$





II. B. Continued) Hawes ⑦

5. Applying this to double integrals

$$y(x) = -\int_a^x \{A(t) + (x-t)[B(t) - A'(t)]\} y(t)\,dt$$

$$+ \int_a^x (x-t) g(t)\,dt + (x-a)[A(a) y_0 + y_0'] + y_0$$

6. Identify: a. $K(x,t) = -\{A(t) + (x-t)[B(t) - A'(t)]\}$

b. $f(x) = \int_a^x (x-t) g(t)\,dt + (x-a)[A(a) y_0 + y_0'] + y_0$

c. NOTE: Here $A(t), B(t), g(t)$, and y_0', y_0 are known!

7. Thus $y(x) = f(x) + \int_a^x K(x,t) y(t)\,dt$

$\Rightarrow$ Volterra Eq of Second Kind

C. Ex: Linear Oscillator Equation

1. a. $y'' + \omega^2 y = 0$

b. Boundary Conditions: $y(0) = 0$, $y'(0) = 1$

2. Here $A(x) = 0$, $B(x) = \omega^2$, $g(x) = 0$

a. $K(x,t) = -\{0 + (x-t)[\omega^2 - 0]\} = \omega^2(t-x)$

Lower limit $a=0$

b. $f(x) = \int_a^x (x-t)(0)\,dt + (x-a)[0(0) + 1] + 0 = (x-a) = x$

3. Thus $\boxed{y(x) = x + \int_0^x \omega^2 (t-x) y(t)\,dt}$

a. Corresponds to original differential equations and incorporates boundary conditions.

D. Integral Equations with Dirichlet Boundary Conditions

1. Consider $y'' + \omega^2 y = 0$ with Dirichlet BCs, $y(0)=0$, $y(b)=0$.
   $\Rightarrow$ Requires modification of procedure since $y'(0)$ is unknown.
2. First integration: $y' = -\omega^2 \int_0^x y\, dt + y'(0)$
3. Second integration: a. $y = -\omega^2 \underbrace{\int_0^x du \int_0^u y\, dt}_{= \int_0^x (x-t) y(t)\, dt} + x y'(0)$

   b. Thus $y(x) = -\omega^2 \int_0^x (x-t) y(t)\, dt + x y'(0)$
4. To eliminate $y'(0)$, impose B.C. at $x=b$ on this result
   a. $y(b) = 0 = -\omega^2 \int_0^b (b-t) y(t)\, dt + b y'(0)$
   b. Thus, $y'(0) = \frac{\omega^2}{b} \int_0^b (b-t)\, y(t)\, dt$
   c. $y(x) = -\omega^2 \int_0^x (x-t) y(t)\, dt + \frac{\omega^2 x}{b} \int_0^b (b-t)\, y(t)\, dt$
5. To simplify break $\int_0^b dt = \int_0^x dt + \int_x^b dt$
   a. $y(x) = \omega^2 \int_0^x \left[\frac{x}{b}(b-t) - (x-t)\right] y(t)\, dt + \omega^2 \int_x^b \frac{x}{b}(b-t)\, y(t)\, dt$

   NOTE: x is not integration variable, so it be brought into the integral!

   b. NOTE: $\frac{x}{b}(b-t) - (x-t) = \cancel{x} - \frac{xt}{b} - \cancel{x} + t = t\left(1 - \frac{x}{b}\right) = \frac{t}{b}(b-x)$
   c. Thus
   $y(x) = \omega^2 \int_0^x \frac{t}{b}(b-x)\, y(t)\, dt + \omega^2 \int_x^b \frac{x}{b}(b-t)\, y(t)\, dt$
6. Define Kernel: $\boxed{K(x,t) = \begin{cases} \frac{t}{b}(b-x) & 0 \le t < x \\ \frac{x}{b}(b-t) & x < t \le b \end{cases}}$

7. Thus $\boxed{y(x) = \omega^2 \int_0^b k(x,t)\, y(t)\, dt}$ eigenvalue ω^2

Homogeneous Fredholm Eigenvalue Equation of Second Kind

E. Some Final Comments

1. Kernel Properties

a. Symmetric $K(x,t) = K(t,x)$

b. Continuous at $t = x$

c. Derivative $\frac{\partial K(x,t)}{\partial t}$ is discontinuous at $t = x$.

2. $K(x,t)$ is a Green's Function of ODE with specified BC's.

3. Boundary Condition determine Type of integral equation

a. Cauchy Boundary Conditions $\Rightarrow$ Volterra Eq.
$y(0)$, $y'(0)$

b. Dirichlet Boundary Conditions $\Rightarrow$ Fredholm Eq.
$y(a)$, $y(b)$