Lecture #22: Hilbert-Schmidt Theory and Introduction to Calculus of Variations

I. Hilbert-Schmidt Theory for Integral Equations

A. Symmetrization of Kernels

1. Hilbert-Schmidt theory applies to Fredholm equations with symmetric kernels: $K(x,t) = K(t,x)$

2. Symmetry leads to a parallel with Sturm-Liouville Theory for ODEs

3. Some nonsymmetric kernels can be symmetrized.
   a. $\phi(x) = f(x) + \lambda \int_a^b K(x,t)\rho(t)\phi(t)\,dt$
   b. Kernel here is $K(x,t)\rho(t) \leftarrow$ not symmetric.
   c. But, multiply by $\sqrt{\rho(x)}$ and substitute $\sqrt{\rho(x)}\,\phi(x) = \psi(x)$ to obtain

$$\psi(x) = \sqrt{\rho(x)}\,f(x) + \lambda \int_a^b \underbrace{\left[K(x,t)\sqrt{\rho(x)\rho(t)}\right]}_{\text{symmetric!}} \psi(t)\,dt$$

B. Properties

1. Consider a homogeneous Fredholm equation of the Second Kind:

$$\phi(x) = \lambda \int_a^b K(x,t)\,\phi(t)\,dt$$

where the Kernel $K(x,t)$ is symmetric and real

2. Parallel to Sturm-Liouville theory for ODEs, it can be proved that:
   a. Eigenvalues λ are real
   b. Eigenfunctions $\phi_n(x)$ are orthogonal and complete

I. B. (Continued)

3.
$$\phi(x) = \lambda \underbrace{\int_a^b K(x,t)\,\phi(t)\,dt}_{\text{Linear operator}}$$

a. $\mathcal{K}\phi(x) \equiv \int_a^b K(x,t)\,\phi(t)\,dt$

b. Thus

$$\mathcal{K}\phi(x) = \frac{1}{\lambda}\phi(x) \leftarrow \text{Eigenvalue problem with eigenvalue } \frac{1}{\lambda}$$

c. Operator $\mathcal{K}$ is <u>linear</u>.

4. <u>Scalar Product</u> $\langle \psi | \phi \rangle \equiv \int_a^b \psi^*(x)\,\phi(x)\,dx$

5. <u>Self-Adjoint operator, $\mathcal{K}$</u>:

a.
$$\langle \psi | \mathcal{K}\phi \rangle = \int_a^b \psi^*(x)[\mathcal{K}\phi(x)]\,dx = \int_a^b \psi^*(x)\left[\int_a^b K(x,t)\,\phi(t)\,dt\right]dx$$
$$= \int_a^b \phi(t)\left[\int_a^b \underbrace{K(x,t)}_{=K^*(t,x)}\psi^*(x)\,dx\right]dt = \int_a^b \mathcal{K}^*\psi^*(t)\,\phi(t)\,dt$$
$$= \langle \mathcal{K}\psi | \phi \rangle$$

b. Self-adjointness of $\mathcal{K}$ means: i) Eigenvalues are real

ii) Eigenvectors are orthogonal (except degeneracies)

6. <u>Proof of Real λ and Orthogonality</u>

a. For two solutions, $\frac{1}{\lambda_i}\phi_i(x) = \int_a^b K(x,t)\,\phi_i(t)\,dt$

and $\frac{1}{\lambda_j}\phi_j(x) = \int_a^b K(x,t)\,\phi_j(t)\,dt$

b. Multiply by ϕ_j^* or ϕ_i^* and combine [exploiting symmetry and reality of $K(x,t)$] to obtain

$$\left(\frac{1}{\lambda_i} - \frac{1}{\lambda_j^*}\right)\int_a^b \phi_j^*(x)\,\phi_i(x)\,dx = 0$$

c. If $i=j$, integral is non zero, so $\boxed{\lambda_i = \lambda_i^*}$ → real eigenvalues.

d. If $\lambda_i \neq \lambda_j$, then $\boxed{\int_a^b \phi_j^*(x)\,\phi_i(x)\,dx = 0}$ → orthogonality of eigenfunctions.

I. B. (Continued)

7. Completeness:

a. Assume orthogonal eigenfunctions have been normalized.

b. For $g(x) = \int K(x,t)\, h(t)\, dt$ ← piecewise, continuous function

we can represent $\boxed{g(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)}$ ← completeness.

8. Express $K(x,t)$ in terms of eigenfunction $\phi_n(x)$:

a. Assume $K(x,t) = \sum_{n=1}^{\infty} a_n(x)\, \phi_n(t)$

b. $\phi_i(x) = \lambda_i \int_a^b K(x,t)\, \phi_i(t)\, dt$

↑ substitute & use orthogonality

c. $\Rightarrow \phi_i(x) = \lambda_i a_i(x)$

d. Thus $\boxed{K(x,t) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\, \phi_n(t)}{\lambda_n}}$ ← Same as eigenfunction expansion of Green's functions

9. NOTE: a. Hilbert-Schmidt theory does not solve for eigenfunctions $\phi_n(x)$ and eigenvalues λ_n, but is used to prove real λ and orthogonal & complete $\phi_n(x)$.

b. To solve for λ_n & $\phi_n(x)$, use methods in previous lecture!

C. Solving Inhomogeneous Integral Equation

1. $\phi(x) = f(x) + \lambda \int_a^b K(x,t)\, \phi(t)\, dt$

2. Assume solutions of homogeneous equation ($f(x)=0$) are known, λ_n, $\phi_n(x)$

3. Expand $\phi(x)$ & $f(x)$ in eigenfunctions:

$\phi(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$ ← unknown

$f(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$ ← known

I. C. Continued)

4. Substituting:
$$\sum_{n=1}^{\infty} a_n \phi_n(x) = \sum_{n=1}^{\infty} b_n \phi_n(x) + \lambda \underbrace{\int_a^b K(x,t) \sum_{n=1}^{\infty} a_n \phi_n(t)\,dt}$$

$$\sum_{n=1}^{\infty} a_n \underbrace{\int_a^b K(x,t)\phi_n(t)\,dt}_{\text{From solution to homogeneous eq.}} = \sum_{n=1}^{\infty} a_n \frac{\phi_n(x)}{\lambda_n}$$

5. Thus, we can multiply by $\phi_i(x)$ and $\int_a^b dx$.

a. Orthogonality will then yield
$$a_i = b_i + \lambda \frac{a_i}{\lambda_i} \overset{\text{Solving}}{\Longrightarrow} a_i = b_i + \frac{\lambda}{\lambda_i - \lambda} b_i$$

6. Therefore our solution is $\phi(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$
$$\boxed{\phi(x) = f(x) + \lambda \sum_{i=1}^{\infty} \frac{\phi_i(x)}{\lambda_i - \lambda} \int_a^b f(t)\phi_i(t)\,dt}$$

a. We can construct inhomogeneous solutions from homogeneous eigenfunctions

II. Example: Inhomogeneous Fredholm Equation

1. $\phi(x) = x^3 + \lambda \int_{-1}^{1} (t+x)\phi(t)\,dt$ (for $\lambda = 1$)

2. We solved the homogeneous equation in Lec#21, I.C.4.

Normalized Form ↓

$$\lambda_1 = \frac{\sqrt{3}}{2} \qquad \phi_1(x) = \frac{\sqrt{3}}{2}\left(x + \frac{1}{\sqrt{3}}\right)$$
$$\lambda_2 = -\frac{\sqrt{3}}{2} \qquad \phi_2(x) = \frac{\sqrt{3}}{2}\left(x - \frac{1}{\sqrt{3}}\right)$$

3. From Hilbert-Schmidt approach, ($\lambda = 1$)

a. $\phi(x) = x^3 + \frac{\phi_1(x)}{\lambda_1 - 1}\int_{-1}^{1} t^3 \phi_1(t)\,dt + \frac{\phi_2(x)}{\lambda_2 - 1}\int_{-1}^{1} t^3 \phi_2(t)\,dt$

b. After some calculation, $\boxed{\phi(x) = x^3 - \frac{6}{5}(2x+1)}$

II. Calculus of Variations

A. General Comments

1. Variational principles — in which a function is varied — are powerful for producing generalized and elegant treatments of physical problems

2. For existing theories:
   - a. Unification of different areas of physics ⇒ energy is key concept.
   - b. Convenient analysis: Lagrange equations
   - c. Elegant treatment of constraints

3. Excellent starting point for new, complex topics in physics.

4. Variational analysis can provide proofs of completeness for Sturm-Liouville & Hilbert-Schmidt theory.

B. The Euler Equation

1. Functionals
   - a. A quantity whose arguments are functions, not just variables.
   - b. $$J[y] = \int_{x_1}^{x_2} f\left(y(x), \frac{dy(x)}{dx}, x\right) dx$$

     square brackets denote J as a functional

     f is a function of three variables: y, $\frac{dy}{dx}$, x
   - c. J has a value dependent on the specific choice of $y(x)$.
   - d. J depends on behavoir of $y(x)$ over $x_1 \leq x \leq x_2$.

2. Goal: Find a continuous & differentiable function $y(x)$ that makes J stationary relative to small changes in y within $x_1 \leq x \leq x_2$

II, B. (Continued)

3. NOTE: Stationary values of J are usually minima or maxima, but could also be saddle points.

4. Simplified Notation:

a. Suppresses arguments (x) and [y]

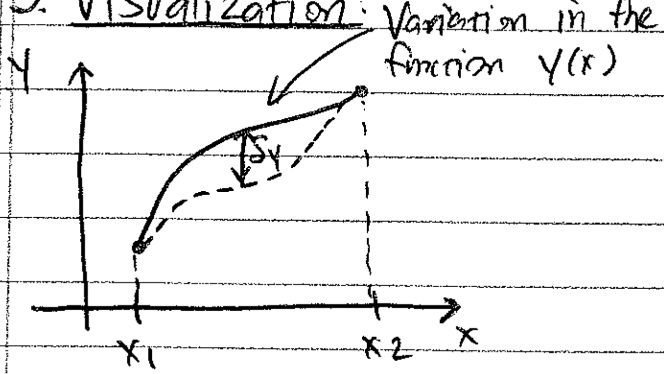
b. Denote $y_x \equiv \frac{dy}{dx}$

c. Variation in J ⇒ δ ← variation of a function.

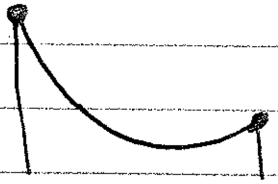
d. Thus, $\boxed{\delta J = \delta \int_{x_1}^{x_2} f(y, y_x, x)\, dx}$

NOTE: y and y_x are treated as independent variables of the function f.

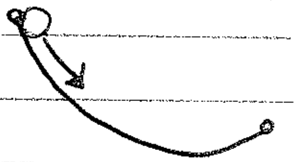
5. Visualization:



6. Example Problems: a. Minimum energy of a rope attached at fixed points in a gravitational field ⇒



b. Track that minimizes travel time between two points subject only to gravity:



(Brachistochrone problem)

7. Consider varying function $y(x)$ with fixed endpoints, $y(x_1)$ and $y(x_2)$
   - a. Deformation of $y(x)$, δy, is described by
     - i. New function $\eta(x)$
     - ii. Scale factor α
   - b. $$\boxed{y(x,\alpha) = \underbrace{y(x,0)}_{\downarrow} + \alpha\,\eta(x)}$$ where $\left.\begin{matrix}\eta(x_1)=0\\ \eta(x_2)=0\end{matrix}\right\}$ fixed endpoints

     Here, $y(x,0)$ is the unknown path that minimizes J.
   - c. Thus, $$\delta y = \alpha\,\eta(x)$$

8. Functional: $$J(\alpha) = \int_{x_1}^{x_2} f(y(x,\alpha), y_x(x,\alpha), x)\,dx$$
   - a. J is now a function of α, not a functional of y
   - b. Stationary Values $$\left[\frac{\partial J(\alpha)}{\partial \alpha}\right]_{\alpha=0} = 0$$

9. Applying $\frac{\partial}{\partial \alpha}$ to the functional:
   - a. $$\frac{\partial J(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y_x}\frac{\partial y_x}{\partial \alpha}\right]dx = 0$$
   - b. NOTE: $$\frac{\partial y(x,\alpha)}{\partial \alpha} = \eta(x)$$ and $$\frac{\partial}{\partial \alpha}\left[y_x(x,\alpha)\right] = \frac{\partial}{\partial \alpha}\left[\frac{dy(x,0)}{dx} + \alpha\frac{d\eta(x)}{dx}\right] = \frac{d\eta(x)}{dx}$$
   - c. Thus $$\frac{\partial J(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y_x}\frac{d\eta(x)}{dx}\right]dx = 0$$
   - d. Integrate 2nd term by parts: $$\int_{x_1}^{x_2}\frac{\partial f}{\partial y_x}\frac{d\eta(x)}{dx}\,dx = \cancelto{0\ \text{endpoints}}{\left[\frac{\partial f}{\partial y_x}\eta\right]_{x_1}^{x_2}} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial y_x}\right)\eta(x)\,dx$$

     $u = \frac{\partial f}{\partial y_x}$ $\quad dv = \frac{d\eta}{dx}dx$

     $du = \frac{d}{dx}\frac{\partial f}{\partial y_x}dx$ $\quad v = \eta$

II. B. (Continued)

10. Thus $\frac{\partial J(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} \right] \eta(x)\, dx = 0$

a. Must be true for arbitrary $\eta(x)$ $\Rightarrow$ $[\cdots]$ must vanish

11. **Euler Equation:** $\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} = 0}$

12. **Properties:**

a. Form of f is known, so reduces to ODE for $y(x)$.

b. Since $f = f(y, y_x, x)$, $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y_x}\frac{d^2y}{dx^2}$

C. **Example: Shortest Distance in Euclidean (x,y) plane**

1. Compute shortest distance between (x_1, y_1) and (x_2, y_2)
2. Element of distance: $ds = [(dx)^2 + (dy)^2]^{\frac{1}{2}} = [1 + (\frac{dy}{dx})^2]^{\frac{1}{2}} dx = (1 + y_x^2)^{\frac{1}{2}} dx$
3. Functional:

$$J = \int_{x_1, y_1}^{x_2, y_2} ds = \int_{x_1}^{x_2} (1 + y_x^2)^{\frac{1}{2}} dx$$

a. Therefore: $f(y, y_x, x) = (1 + y_x^2)^{\frac{1}{2}}$

4. Euler Equation: $\left[\cancelto{0}{\frac{\partial f}{\partial y}} - \frac{d}{dx}\frac{\partial f}{\partial y_x} \right] = -\frac{d}{dx}\left[\frac{\partial f}{\partial y_x}\right] = 0$

a. $\frac{\partial f}{\partial y_x} = \frac{1}{2}(1 + y_x^2)^{-\frac{1}{2}}(2y_x) = \frac{y_x}{(1 + y_x^2)^{\frac{1}{2}}}$

5. $\frac{d}{dx}\left[\frac{y_x}{(1 + y_x^2)^{\frac{1}{2}}}\right] = \frac{1}{(1 + y_x)^{\frac{1}{2}}}\frac{dy_x}{dx} - \frac{1}{2}\frac{y_x(2y_x)}{(1 + y_x^2)^{\frac{3}{2}}}\frac{dy_x}{dx} = \frac{1}{(1 + y_x^2)^{\frac{3}{2}}}\overset{= \frac{dy_x}{dx}}{\frac{d^2y}{dx^2}} = 0$

6. This requires $\boxed{\frac{d^2y}{dx^2} = 0}$. Integrating twice, we can obtain the equation for a straight line $\boxed{y = mx + b}$ where m & b are chosen to satisfy endpoints.