# Lecture #23: Calculus of Variations & Hamilton's Equations

## I. Calculus of Variations

### A. Review:

1. Variation of a functional: $\delta J[y] = \delta \int_{x_1}^{x_2} f(y, y_x, x)\, dx = 0$ $\quad y_x \equiv \frac{dy}{dx}$

2. Take $y(x,\alpha) = \underbrace{y(x,0)}_{\text{minimizes } J} + \alpha\, \eta(x)$ ← Variation

3 $\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[ \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} \right] \eta(x)\, dx = 0$

4. Euler Equation: $\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} = 0}$

### B. Ex: Optical Path near a Black Hole

1. Velocity of light $v(y) = \frac{y}{b}$ where $b > 0$

   a. At $y=0 \rightarrow v=0 \Rightarrow$ event horizon

2. Fermat's Principle: Light will take path of shortest travel time.

$$\Delta t = \int dt = \int_{x_1,y_1}^{x_2,y_2} \frac{ds}{v} = \int \frac{b}{y} ds = b\int_{x_1,y_1}^{x_2,y_2} \frac{\sqrt{dx^2+dy^2}}{y} \leftarrow \text{minimize}$$

3. We are free to choose $x$ or $y$ as the independent variable.

   a. Using $y$ is easier here: $\Rightarrow dy\sqrt{\frac{dx^2}{dy^2}+1}$

   b. $\Delta t = \int_{y_1}^{y_2} \frac{\sqrt{x_y^2+1}}{y}\, dy$

4. Euler Eq: $f(x, x_y, y) = \frac{\sqrt{x_y^2+1}}{y} \Rightarrow \cancel{\frac{\partial f}{\partial x}}^{\,0} - \frac{d}{dy}\frac{\partial f}{\partial x_y} = 0$

   a. $\frac{\partial f}{\partial x_y} = \frac{x_y}{y\sqrt{x_y^2+1}}$

I. B4. (Continued)

b. Thus, $\frac{d}{dy}\left[\frac{x_y}{y\sqrt{x_y^2+1}}\right]=0$

c. Integrating $\int_0^y dy$ yields $\frac{x_y}{y\sqrt{x_y^2+1}} = C_1 \leftarrow$ constant

5. To solve this, solve for $x_y$: $x_y = \frac{dx}{dy} = \frac{C_1 y}{\sqrt{1-C_1^2y^2}}$

a. Separating variables $\int^x dx = \int^y dy \frac{C_1 y}{\sqrt{1-C_1^2y^2}} = \frac{-(1-C_1^2y_1^2)^{\frac{1}{2}}}{C_1}$

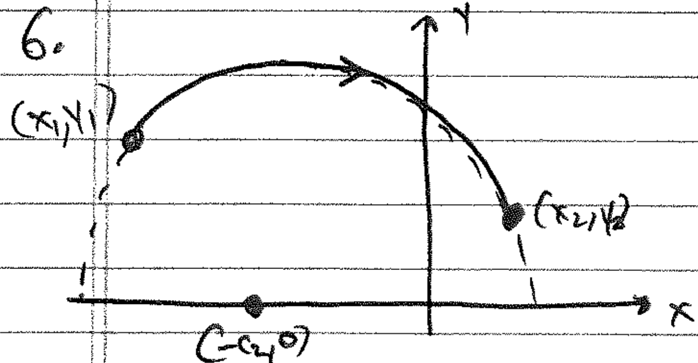
$$\frac{d}{dy}(1-C_1^2y_1^2)^{\frac{1}{2}} = \frac{1}{2}\frac{(-C_1^2 2y_1)}{(1-C_1^2y_1^2)^{\frac{1}{2}}} = C_1 \frac{-C_1y_1}{\sqrt{1-C_1^2y_1^2}}$$

b. Thus

$$x+C_2 = -\frac{\sqrt{1-C_1^2y^2}}{C_1} \Rightarrow \boxed{(x+C_2)^2+y^2 = \frac{1}{C_1^2}}$$

Equation of a circle centered at $(-C_2, 0)$ with radius $C_1^{-1}$!

6.




## C. Alternate Forms of Euler Equations

1. Equivalent form: $\boxed{\frac{\partial f}{\partial x} - \frac{d}{dx}\left[f - y_x\frac{\partial f}{\partial y_x}\right]=0}$

using $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y_x}\frac{d^2y}{dx^2}$

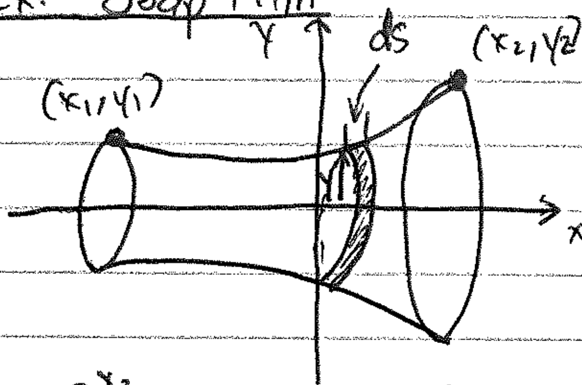
2. If $f=f(y, y_x)$ (no explicit dependence on $x$), then

$$\boxed{f - y_x\frac{\partial f}{\partial y_x} = \text{constant}}$$

I. (Continued)

D. Ex: Soap Film

1.




a. Minimize area of surface of revolution of $y(x)$.

b. $dA = 2\pi y\, ds = 2\pi y (dx^2 + dy^2)^{\frac{1}{2}}$

$$= 2\pi y (1 + y_x^2)^{\frac{1}{2}} dx$$

2. $$J = \int_{x_1}^{x_2} 2\pi y (1 + y_x^2)^{\frac{1}{2}} dx$$

a. Neglecting constant $2\pi$, $f(y, y_x, x) = y(1 + y_x^2)^{\frac{1}{2}}$

3. Since $f$ does not depend explicitly on $x$, use alternate form

$$f - y_x \frac{\partial f}{\partial y_x} = c_1 \leftarrow \text{constant}$$

4. a. $$\frac{\partial f}{\partial y_x} = \frac{\frac{1}{2} y \cancel{2} y_x}{(1 + y_x^2)^{\frac{1}{2}}}$$

b. Thus, $$y(1 + y_x^2)^{\frac{1}{2}} - y_x \frac{y\, y_x}{(1 + y_x^2)^{\frac{1}{2}}} = c_1$$

5. Simplifying

a. $$\frac{y}{(1 + y_x^2)^{\frac{1}{2}}} \left[ 1 + \cancel{y_x^2} - \cancel{y_x^2} \right] = \frac{y}{(1 + y_x^2)^{\frac{1}{2}}} = c_1$$

b. $$y^2 = c_1^2 (1 + y_x^2) \Rightarrow \frac{y^2 - c_1^2}{c_1^2} = y_x^2$$

c. I can use $y$ instead $x$ as independent variable: $(y_x)^{-1} = \frac{dx}{dy}$, so

$$(y_x)^{-1} = \frac{c_1}{\sqrt{y^2 - c_1^2}} \Rightarrow \frac{dx}{dy} = \frac{c_1}{\sqrt{y^2 - c_1^2}}$$

I. D. (Continued)

6. NOTE: $\int \frac{dx}{\sqrt{x^2-a^2}} = a\cosh^{-1}\left(\frac{x}{a}\right)$ ~~+ constant~~ , so

a. $x = c_1 \cosh^{-1}\left(\frac{y}{c_1}\right) + c_2$

b. Solving for y: $\boxed{y = c_1 \cosh\left(\frac{x-c_2}{c_1}\right)}$ ← Catenoid (Catenary of revolution)

7. <u>NOTE:</u> Value of $c_1$ must be small enough that $\sqrt{y^2-c_1^2}$ is real!

a. Analyze physical problem to be sure answer makes sense!
→ Need a differentiable solution.

b. If $c_1$ is too large, soap solution will break → discontinuous.

# II. Hamilton's Equations

## A. Generalization of Euler Equation to Many Dependent Variables

1. $J = \int_{x_1}^{x_2} f(u_1(x), u_2(x), \ldots, u_n(x), u_{1x}(x), u_{2x}(x), \ldots u_{nx}(x), x)\,dx$

a. $u_{i_x}(x) \equiv \frac{du_i(x)}{dx}$

b. $u_i(x,\alpha) = u_i(x,0) + \alpha\, \eta_i(x) \qquad i = 1, 2, \ldots, n$

c. $\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \sum_i \left(\frac{\partial f}{\partial u_i}\eta_i + \frac{\partial f}{\partial u_{ix}}\eta_{ix}\right)dx \underset{\uparrow}{=} \int_{x_1}^{x_2} \sum_i \left(\frac{\partial f}{\partial u_i} - \frac{d}{dx}\frac{\partial f}{\partial u_{ix}}\right)\eta_i\,dx$

integrate by parts

d. Thus $\boxed{\frac{\partial f}{\partial u_i} - \frac{d}{dx}\frac{\partial f}{\partial u_{ix}} = 0} \qquad i = 1, 2, \ldots, n \qquad$ n Euler Eq.'s

II. (Continued)

B. Hamilton's Principle

1. Nonrelativistic Lagrangian $\boxed{L \equiv \underset{\text{Kinetic energy}}{T} - \underset{\text{potential energy}}{V}}$

2a. Chose time as independent variable $x \to t$

b. Chose $x_i$ as dependant variables $y_i \to x_i(t)$, $y_{ix} \to \dot{x}_i(t)$

3. Hamilton's Principle

> The motion of a system from $t_1$ to $t_2$ yields a stationary value of the action (the time integral of the Lagrangian).

$$\delta J[x_i] = \delta \int_{t_1}^{t_2} L(x_1, x_2, \ldots x_n, \dot{x}_1, \dot{x}_2, \ldots \dot{x}_n, t)\, dt = 0$$

4. Lagrangian Equations of Motion: $\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0}$

5. Advantages of Lagrangian formulation

a. Involves only scalar quantities

b. Invariant with respect to choice of coordinate system

c. Using energy as organizing principle, can extend Lagrangian formulation to diverse systems, electrical systems, relativistic sys, etc

d. Unification of separate areas of physics

e. Shows relation by symmetries & conservation Laws

i. For ignorable coordinate $x_i$ (Lagrangian independent of $x_i$)

$\Rightarrow$ Conservation of momentum conjugate to $x_i$

$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0$

II.B. (Continued)

6. Example: Particle Motion in Cartesian Coordinates

a. Mass $m$ moving in one dimension $x$, potential $V(x)$

b. $T = \frac{1}{2} m \dot{x}^2$

c. $L = \frac{1}{2} m \dot{x}^2 - V(x)$

d. $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$ $\quad \frac{\partial L}{\partial x} = -\frac{\partial V(x)}{\partial x} \equiv F(x)$

e. $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0 \Rightarrow \boxed{\frac{d}{dt}(m\dot{x}) - F(x) = 0}$ Newton's 2nd Law

C. Hamilton's Equations

1. Def: Canonical Momentum: $\boxed{p_i = \frac{\partial L}{\partial \dot{q}_i}}$

a. $T = \frac{1}{2} m \dot{q}^2$, $p_i = m\dot{q}_i$

Lagrangian eq

$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt} p_i = \dot{p}_i$

2. $dL = \sum_i \left( \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \right) + \frac{\partial L}{\partial t} dt = \sum_i (\dot{p}_i dq_i + p_i d\dot{q}_i) + \frac{\partial L}{\partial t} dt$

3. Def: Hamiltonian: $\boxed{H \equiv \sum_i p_i \dot{q}_i - L}$

4. $dH = \sum_i (\dot{q}_i dp_i + \dot{p}_i dq_i) - \frac{\partial L}{\partial t} dt$

$= \sum_i \left( \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i \right) + \frac{\partial H}{\partial t} dt$

Equating coefficients of $dp_i, dq_i, dt$

5. Hamilton's Equations: $\boxed{\frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$

D. Several Independent Variables

1. $U(x,y,z) \Rightarrow J = \iiint f(U, U_x, U_y, U_z, x, y, z)\, dx\, dy\, dz$

a. $U(x,y,z,\alpha) = U(x,y,z,0) + \alpha \eta(x,y,z)$

II. D. (Continued)

2. Euler Equation: $$\boxed{\frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial u_y} - \frac{\partial}{\partial z}\frac{\partial f}{\partial u_z} = 0}$$

a. <u>NOTE</u>: $\frac{\partial}{\partial x}$ (& $\frac{\partial}{\partial y}, \frac{\partial}{\partial z}$) above are <u>not</u> usual partial derivatives, but depend on <u>all</u> x dependence of $\frac{\partial f}{\partial u_x}$.

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u_x}\right) = \frac{\partial^2 f}{\partial u_x \partial x} + \frac{\partial^2 f}{\partial u_x \partial u}\frac{du}{dx} + \frac{\partial^2 f}{\partial u_x^2}\frac{d^2u}{dx^2}$$

3. <u>Ex: Laplace's Equation</u>

a. Electrostatics: energy density $= \frac{1}{2}\varepsilon|\underset{\sim}{E}|^2 = \frac{1}{2}\varepsilon|\nabla\phi|^2$, $\phi(x,y,z)$

b. Impose requirement the electrostatic energy is minimized in a charge-free volume (subject to B.C.'s on $\phi$).

c. $$J = \iiint |\nabla\phi|^2 \, dx\,dy\,dz = \iiint (\phi_x^2 + \phi_y^2 + \phi_z^2)\, dx\,dy\,dz$$

d. Thus $f = \phi_x^2 + \phi_y^2 + \phi_z^2$

e. Euler's Eq: $$\cancel{\frac{\partial f}{\partial \phi}}^{\,0} - \frac{\partial}{\partial x}\frac{\partial f}{\partial \phi_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial \phi_y} - \frac{\partial}{\partial z}\frac{\partial f}{\partial \phi_z} = 0$$

$$\Rightarrow -2(\phi_{xx} + \phi_{yy} + \phi_{zz}) = 0$$

$$\Rightarrow \boxed{\nabla^2\phi(x,y,z) = 0} \leftarrow \text{Laplace's Equation}$$

E. <u>Several Dependent & Independent Variables</u>

1. $y_i(x_j)$ $\quad i = 1, 2, \ldots n \quad j = 1, 2, \ldots m$

2. Generalization: $$\boxed{\frac{\partial f}{\partial y_i} - \sum_j \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial y_{ij}}\right) = 0}, \quad y_{ij} \equiv \frac{\partial y_i}{\partial x_j}$$

$i = 1, 2, \ldots n$ dependent variables.

F. <u>Development for New Areas of Physics</u>

1. If the basic physics is not yet known, a postulated variational principle can be a useful starting point.