Lecture #24: Lagrangian Multipliers

I. Using Lagrangian Multipliers for Constrained Minima/Maxima

A. Constrained Optimization

1. Consider minimization of a function $f(x,y,z)$ subject to the constraint $g(x,y,z) = C$.

2. Since $g(x,y,z) = C$ defines a surface, one wants to minimize $f(x,y,z)$ on that surface.

3. In principle, one could solve for $z = z(x,y)$ on the surface, and minimize using $\frac{\partial}{\partial x}\left[f(x,y,z(x,y))\right] = 0$ & $\frac{\partial}{\partial y}\left[f(x,y,z(x,y))\right] = 0$

   a. But this method is not always possible

   b. Also, variables x, y & z are not treated on an equivalent basis.

4. Method of Lagrangian Multipliers is an alternative

B. Lagrangian Multipliers

1. For $g(x,y,z) = C \leftarrow$ constant, we have

$$dg = \left(\frac{\partial g}{\partial x}\right)_{yz} dx + \left(\frac{\partial g}{\partial y}\right)_{xz} dy + \left(\frac{\partial g}{\partial z}\right)_{xy} dz = 0$$

2. For $dy = 0$ (y held constant), we have

$$\left(\frac{\partial g}{\partial x}\right)_{yz} dx = -\left(\frac{\partial g}{\partial z}\right)_{xy} dz \Rightarrow \boxed{\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\left(\frac{\partial g}{\partial x}\right)_{yz}}{\left(\frac{\partial g}{\partial z}\right)_{xy}}}$$

3. Similarly, for $dx = 0$, $\left(\frac{\partial z}{\partial y}\right)_x = -\frac{\left(\frac{\partial g}{\partial y}\right)_{xz}}{\left(\frac{\partial g}{\partial z}\right)_{xy}}$

I. B. (Continued)

4. We want to find $\left(\frac{\partial f}{\partial x}\right)_y = 0$

a. $\left(\frac{\partial f}{\partial x}\right)_y = \left(\frac{\partial f}{\partial x}\right)_{yz} + \left(\frac{\partial f}{\partial z}\right)_{xy} \underbrace{\left(\frac{\partial z}{\partial x}\right)_y}_{= -\frac{\left(\frac{\partial g}{\partial x}\right)_{yz}}{\left(\frac{\partial g}{\partial z}\right)_{xy}}} = \left(\frac{\partial f}{\partial x}\right)_{yz} - \underbrace{\frac{\left(\frac{\partial f}{\partial z}\right)_{xy}}{\left(\frac{\partial g}{\partial z}\right)_{xy}}}_{=\lambda,\ \text{Lagrangian multiplier}} \left(\frac{\partial g}{\partial x}\right)_{yz} = 0$

b. Thus,

$$\boxed{\left(\frac{\partial f}{\partial x}\right)_{yz} - \lambda\left(\frac{\partial g}{\partial x}\right)_{yz} = 0 \quad \text{with} \quad \lambda \equiv \frac{\left(\frac{\partial f}{\partial z}\right)_{xy}}{\left(\frac{\partial g}{\partial z}\right)_{xy}}}$$

5. Similarly,

$$\boxed{\begin{gathered}\left(\frac{\partial f}{\partial y}\right)_{xz} - \lambda\left(\frac{\partial g}{\partial y}\right)_{xz} = 0 \\ \left(\frac{\partial f}{\partial z}\right)_{xy} - \lambda\left(\frac{\partial g}{\partial z}\right)_{xy} = 0\end{gathered}}$$

6. <u>Generalization:</u> n variables & k constraints

$$\boxed{\frac{\partial f}{\partial x_i} - \sum_{j=1}^{k} \lambda_j \frac{\partial g_j}{\partial x_i} = 0 \qquad i = 1, 2, \ldots, n}$$

7. <u>NOTE:</u> This procedure finds minima, maxima, and saddle points so <u>you must subsequently determine the nature of the stationary point.</u>

C. <u>Example: Minimizing Surface-to-Volume Ratio</u>

1. For a right cylinder of radius r & height h, what ratio $\frac{h}{r}$ minimizes the surface area for a fixed volume.

2. $\underbrace{S = 2\pi(rh + r^2)}_{\text{minimize this function, } S(r,h)}$ $\qquad \underbrace{V = \pi r^2 h}_{\text{Constraint, } V(r,h)}$

I. C. (Continued)

3. Constrained equations:

a. $\frac{\partial S}{\partial r} - \lambda \frac{\partial V}{\partial r} = 0$

b. $\frac{\partial S}{\partial h} - \lambda \frac{\partial V}{\partial h} = 0$

4. Thus

a. $2\pi(h+2r) - \lambda 2\pi r h = 0$

b. $2\pi r - \lambda \pi r^2 = 0$

5. Eliminate λ: a. from 4b., $\lambda = \frac{2}{r}$

b. Substituting into 4a. $2\pi h + 4\pi r - \left(\frac{2}{r}\right) 2\pi r h = 0$

$4\pi r - 2\pi h = 0 \Rightarrow \boxed{\frac{h}{r} = 2}$

II. Variation with Constraints

A. Applying Constraints to Functional Variation

1. Functional $J = \int f\left(y_i, \frac{\partial y_i}{\partial x_j}, x_j\right) dx_j$

$x_j \equiv$ Set of independent variables

$y_i \equiv$ Set of dependent variables

2. Apply constraints so that y_i are no longer independent $\Rightarrow$ thus $\eta_i(x_j)$ are also not independent!

3. Constraint: $\phi_k\left(y_i, \frac{\partial y_i}{\partial x_j}, x_j\right) = 0$

a. Can be stated in integral form: $\int \underbrace{\lambda_k(x_j)}_{\lambda_k \text{ may be a function of } x_j!} \phi_k\left(y_i, \frac{\partial y_i}{\partial x_j}, x_j\right) dx_j = 0$

b. Satisfied if

$$\underset{\text{Variation}}{\delta} \int \lambda_k(x_j)\, \phi_k\left(y_i, \frac{\partial y_i}{\partial x_j}, x_j\right) dx_j = 0$$

4. If constraint is an integral, $\int \phi_k\left(y_i, \frac{dy_i}{dx_j}, x_j\right) dx_j = \text{Constant}$, then we have $\delta \int \lambda_k\, \phi_k\left(y_i, \frac{\partial y_i}{\partial x_j}, x_j\right) dx_j = 0$

Here λ_k is a constant.

5. Therefore, our constrained variational problem may be written

$$\boxed{\delta \int \left[f\left(y_i, \frac{\partial y_i}{\partial x_j}, x_j\right) + \sum_k \lambda_k \phi_k\left(y_i, \frac{\partial y_i}{\partial x_j}, x_j\right) \right] dx_j = 0}$$

a. Here λ_k <u>may</u> be a function of x_j depending on form of original constraint.

b. Essentially, we treat the entire integrand as a new function to be varied,

$$g\left(y_i, \frac{dy_i}{\partial x_j}, x_j\right) = f + \sum_k \lambda_k \phi_k$$

B. <u>Lagrangian Formulation</u>

1. Without constraints, we have $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_i} - \dfrac{\partial L}{\partial q_i} = 0$

2. With constraints, Hamilton's Principle becomes

$$\delta \int \left[L(q_i, \dot{q}_i, t) + \sum_k \lambda_k(t)\, \phi_k(q_i, t) \right] dt = 0$$

a. Here, the constraint $\phi_k(q_i, t)$ is usually independent of $\dot{q}_i$.

3. Constrained Lagrangian Equations of Motion

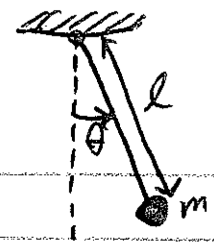
$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_k a_{ik} \lambda_k}$$

where $\boxed{a_{ik} = \dfrac{\partial \phi_k}{\partial q_i}}$

a. Here $\lambda_k a_{ik}$ represents a force in the q_i direction, analogous to $-\dfrac{\partial V}{\partial q_i}$

II. (Continued)

C. Example: Simple Pendulum



1. Simple Pendulum a. mass m
   b. wire of length ℓ

2. Constraint: $\boxed{\phi_1 = r - \ell = 0}$ $\leftarrow r = \ell$

3. $T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$ $\quad V = -mgr\cos\theta \leftarrow V(\frac{\pi}{2}) = 0$

4. $L = T - V = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta$

5. Coefficients: $a_{r1} = \frac{\partial\phi_1}{\partial r} = 1 \quad a_{\theta 1} = \frac{\partial \phi_1}{\partial\theta} = 0$

6. Lagrangian Equations (Constrained)
   a. $\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \lambda_1$
   b. $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$

7. Thus, a. $\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 - mg\cos\theta = \lambda_1$ (units of force in this problem)
   b. $\frac{d}{dt}(mr^2\dot{\theta}) + mgr\sin\theta = 0$ $\Rightarrow$ radial force provided by wire!

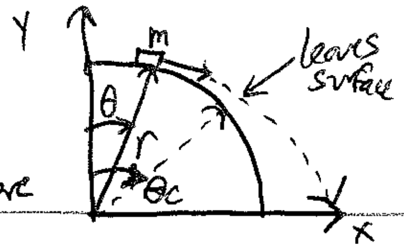
8. Apply constraint to simplify: $r = \ell \Rightarrow \dot{r} = 0$.
   a. $m\ell\dot{\theta}^2 + mg\cos\theta = -\lambda_1$
   b. $m\ell^2\ddot{\theta} + mg\ell\sin\theta = 0$

9. For small amplitude oscillations, $\theta \ll 1$, $\sin\theta \simeq \theta$, so we obtain $\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0 \Rightarrow \theta(t) = A\cos\omega t + B\sin\omega t$, $\omega = \sqrt{\frac{g}{\ell}}$

10. λ_1 represents tension in wire
    a. Depends on $\dot{\theta}(t)$ & $\theta(t)$, so $\lambda_1(t)$ is a function of time

II. (Continued)

D. Example Sliding off a Log



1. What is the maximum angle θ_c before the mass slides off the log?

2. $T = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2)$ $\qquad V = mgr\cos\theta$

3. Constraint: $\phi_1 = r - l = 0$ (valid until mass leaves surface)

   a. We want to find point at which radial force constraint $\to 0$!

4. $L = T - V = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) - mgr\cos\theta$

5. Coefficients $\quad a_{r1} = \frac{\partial\phi_1}{\partial r} = 1 \qquad a_{\theta 1} = \frac{\partial\phi_1}{\partial\theta} = 0$

6. $\frac{d}{dt}\frac{\partial L}{\partial\dot r} - \frac{\partial L}{\partial r} = \lambda_1 \;\Rightarrow\; m\ddot r - mr\dot\theta^2 + mg\cos\theta = \lambda_1(\theta)$ function of time (or θ)

   $\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = 0 \;\Rightarrow\; mr^2\ddot\theta + 2mr\dot r\dot\theta - mgr\sin\theta = 0$

7. Apply constraint $r = l \Rightarrow \dot r = 0, \ddot r = 0$

   a. $-ml\dot\theta^2 + mg\cos\theta = \lambda_1(\theta)$

   b. $ml^2\ddot\theta - mgl\sin\theta = 0$

8. We want to solve for the value of $\theta = \theta_c$ that makes $\lambda_1(\theta_c) = 0$.

   a. $\frac{d}{dt}(7.a.)$ where $\frac{df(\theta)}{dt} = \frac{df(\theta)}{d\theta}\frac{d\theta}{dt} = \frac{df(\theta)}{d\theta}\dot\theta$

   $\Rightarrow -ml2\cancel{\dot\theta}\ddot\theta - mg\sin\theta\cancel{\dot\theta} = \frac{d\lambda_1}{d\theta}\cancel{\dot\theta} \;\Rightarrow\; -ml2\left(\frac{g}{l}\right)\sin\theta - mg\sin\theta = \frac{d\lambda_1}{d\theta}$

   $\ddot\theta = \frac{g}{l}\sin\theta$

   b. Thus, $\frac{d\lambda_1}{d\theta} = -3mg\sin\theta$

   c. Integrating $\int d\theta$, $\quad \lambda_1(\theta) = 3mg\cos\theta + C$

II. D. (Continued) Hawes (7)

9. To fix constant C; from 7a. with $\theta=0$,

a. $-m\ell\dot{\theta}^2\big|_{\theta=0} + mg = \lambda_1(0) = 3mg + C$

b. For $\dot{\theta}(0)=0$ (zero initial velocity), $C = -2mg$.

c. Thus $\boxed{\lambda_1(\theta) = mg(3\cos\theta - 2)}$

10. Now, find θ_c such that $\lambda_1(\theta_c)=0$:

a. $0 = mg(3\cos\theta_c - 2) \Rightarrow \cos\theta_c = \frac{2}{3}$ or $\boxed{\theta_c \simeq 48°}$