Lecture #5 Evaluation of Sums and Other Topics in Complex Analysis

I. Evaluation of Sums

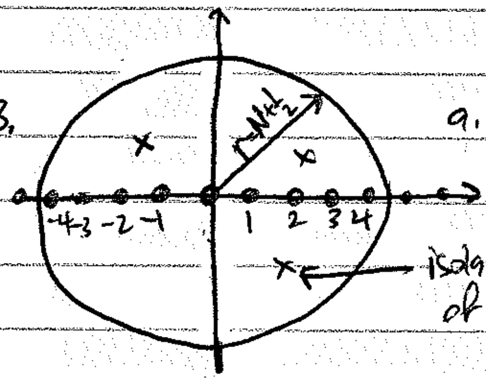
A. Infinite Sums

1. Contour integration can be used to evaluate infinite sums using a trick due to regularly spaced poles associated with tan, cot, sec, csc.

2. Consider poles of $\pi \cot(\pi z)$

a. $\pi \dfrac{\cos(\pi z)}{\sin(\pi z)}$ ← poles when $\sin(\pi z) = 0 \Rightarrow z = n$, n any integer

b. $\operatorname*{Res}_{z=n} \dfrac{\pi\cos(\pi z)}{\sin(\pi z)} = \lim_{z\to n} \dfrac{(z-n)\pi\cos(\pi z)}{\sin(\pi z)} \overset{\text{L'Hopital's Rule}}{=} \lim_{z\to n} \dfrac{-\cancel{(z-n)\pi^2\sin\pi z}^{\,0} + \pi\cos\pi z}{\pi\cos\pi z} = 1$

3. 

a. $I_N = \oint f(z)\,\pi\cot(\pi z)\,dz$

b. By Residue Thm, $I_N = 2\pi i \sum_{n=-N}^{N} f(n) + 2\pi i \sum_k \operatorname*{Res}_{z=z_k}\left[f(z)\,\pi\cot(\pi z)\right]$ ← poles of $f(z)$

4. a. If $\lim_{|z|\to\infty} z f(z) = 0$, then $\lim_{N\to\infty} I_N = 0$ because the integrand on the entire contour becomes negligible (see HW).

b. Thus, $$\boxed{\sum_{n=-\infty}^{\infty} f(n) = -\sum_k \operatorname*{Res}_{z=z_k}\left[f(z)\,\pi\cot(\pi z)\right]}$$

Infinite Sum is given by sum of residues at poles of $f(z)$!

I. A. (Continued)

5. Ex: $S = \sum_{n=1}^{\infty} \frac{1}{n^2+a^2}$

a. First, convert to sum from $n=-\infty$ to ∞

i. NOTE: $S = \sum_{n=-\infty}^{-1} \frac{1}{n^2+a^2}$

ii. Thus $\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \sum_{n=-\infty}^{-1} \frac{1}{n^2+a^2} + \underbrace{\frac{1}{a^2}}_{n=0 \text{ term}} + \sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = 2S + \frac{1}{a^2}$

b. Next, consider $f(z) = \frac{1}{z^2+a^2}$

i. Check that $\lim_{|z|\to\infty} z f(z) = \lim_{|z|\to\infty} \frac{z}{z^2+a^2} = 0$ ✓

ii. Poles of $f(z)$ at $z = \pm ia$

iii. Find Residues of $f(z)\,\pi\cot(\pi z)$ at $z = \pm ia$

a) $\operatorname*{Res}_{z=+ia}\left[\frac{\pi\cot(\pi z)}{(z+ia)(z-ia)}\right] = \frac{\pi\cot(i\pi a)}{2ia} = \boxed{\frac{-\pi\coth(\pi a)}{2a}}$

b) NOTE: $\sin i\overset{\text{real}}{c} = \frac{e^{i(ic)} - e^{-i(ic)}}{2i} = \frac{e^{-c} - e^{+c}}{2i} = +i\,\frac{e^{c} - e^{-c}}{2} = i\sinh c$

$\cos ic = \frac{e^{i(ic)} + e^{-i(ic)}}{2} = \frac{e^{-c} + e^{+c}}{2} = \cosh c$

$\cot ic = \frac{\cos ic}{\sin ic} = \frac{\cosh c}{i\sinh c} = -i\coth c$

c) $\operatorname*{Res}_{z=-ia}\left[\frac{\pi\cot(\pi z)}{(z+ia)(z-ia)}\right] = \frac{\pi\cot(-i\pi a)}{-2ia} = \frac{+\pi\coth(+\pi a)}{+2a}$

c. Thus $2S + \frac{1}{a^2} = 2\left[\frac{\pi\coth(\pi a)}{2a}\right]$

$$\boxed{S = \frac{\pi\coth(\pi a)}{2a} - \frac{1}{2a^2}}$$

I. (Continued)

B. Other Functions Useful for Sums

		Poles	Sum	Formula
1.	$\pi \csc(\pi z)$	$z=n$	$\sum_{n=-\infty}^{\infty} (-1)^n f(n)$	$-\sum_k \operatorname{Res}_{z=z_k}[f(z)\pi\csc(\pi z)]$
	$\pi \tan(\pi z)$	$z=\frac{2n+1}{2}$	$\sum_{n=-\infty}^{\infty} f\left(\frac{2n+1}{2}\right)$	$\sum_k \operatorname{Res}_{z=z_k}[f(z)\pi\tan(\pi z)]$
	$\pi \sec(\pi z)$	$z=\frac{2n+1}{2}$	$\sum_{n=-\infty}^{\infty} (-1)^n f\left(\frac{2n+1}{2}\right)$	$\sum_k \operatorname{Res}_{z=z_k}[f(z)\pi\sec(\pi z)]$

2. If poles of $f(z)$ are at same positions (integral or half-integral values) as poles of trig function used, then treat those as second-order poles.

II. Miscellaneous Topics in Complex Analysis

A. Schwarz Reflection Principle

1. Def: If $f(z)$ is

(1) Analytic over some region including a portion of real axis

(2) real when z is real

then $$f^*(z) = f(z^*)$$

2. Proof:

a. Note: $g^*(z) = \left[(z-x_0)^n\right]^* = (z^*-x_0)^n = g(z^*)$

when x_0 is real and n is an integer

b. $f(z) = \sum_{n=0}^{\infty} (z-x_0)^n \frac{f^{(n)}(x_0)}{n!}$ Taylor expansion about real x_0.

i. Since $f(z)$ is real when z is real, $f^{(n)}(x_0)$ must be real. $\Rightarrow$ Result follows.

II. (Continued)

B. <u>Conformal Mapping</u>:

1. Consider a complex function $f(z) = U(x,y) + iV(x,y)$

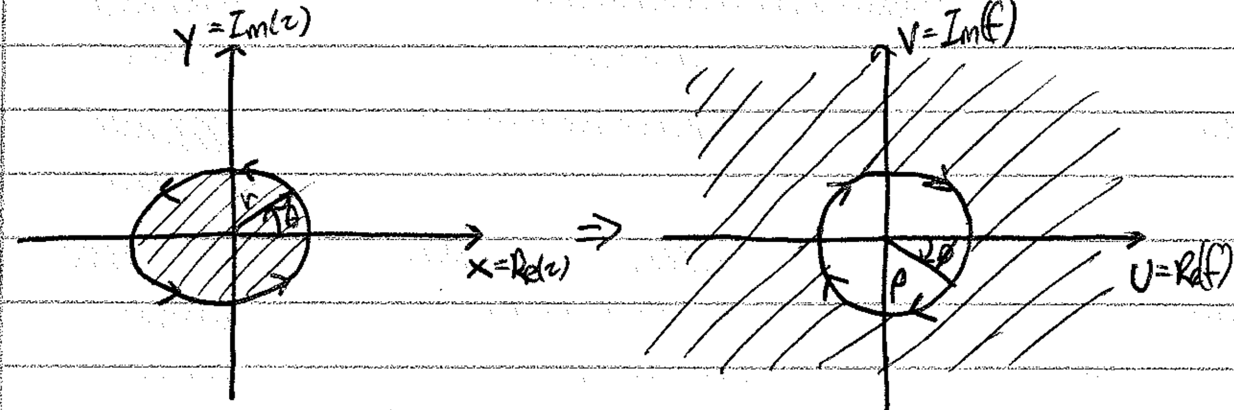
   a. This function represents a mapping from complex (x,y) space to complex (U,V) space.

2. <u>Ex</u>: $f(z) = \frac{1}{z}$

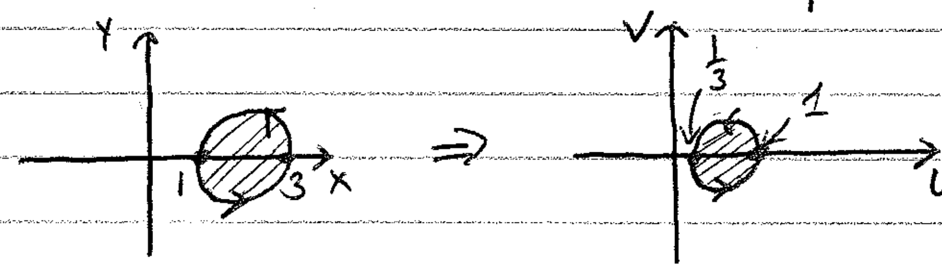
   a. Take $z = re^{i\theta}$ and $f = \rho e^{i\phi}$

      i) $\frac{1}{re^{i\theta}} = r^{-1}e^{-i\theta} = \rho e^{i\phi} \Rightarrow \begin{cases} \rho = r^{-1} \\ \phi = -\theta \end{cases}$

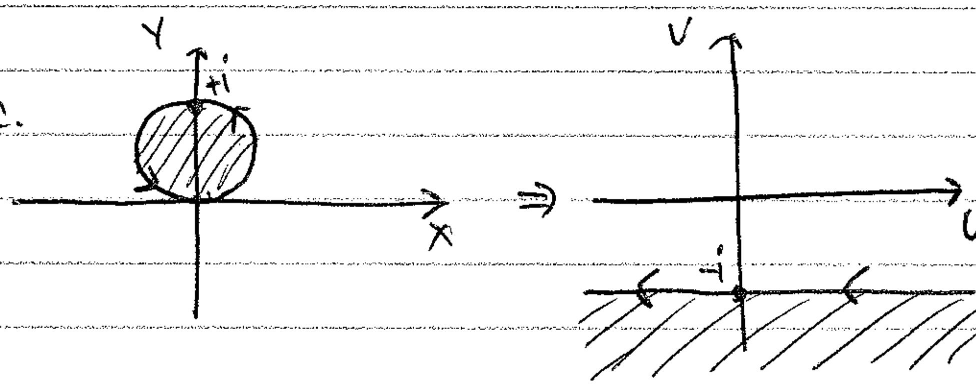
$z = x + iy$



b.



c.



3. Useful for 2D electrostatics and fluid dynamics problems.

IV.B. (Continued)

4. NOTE: Conformal: Except at singularities, preserves angles at which curves intersect.

a. Ex: equipotentials and field lines remain orthogonal!

5. Ex: Stability in Kinetic Plasma.

a. Complex Dispersion Relation $D(p)=0$ yields solutions.

i. $p = -i\omega$ (complex), so if $\omega = \omega_r + i\gamma$

$Im(p) = -\omega_r$

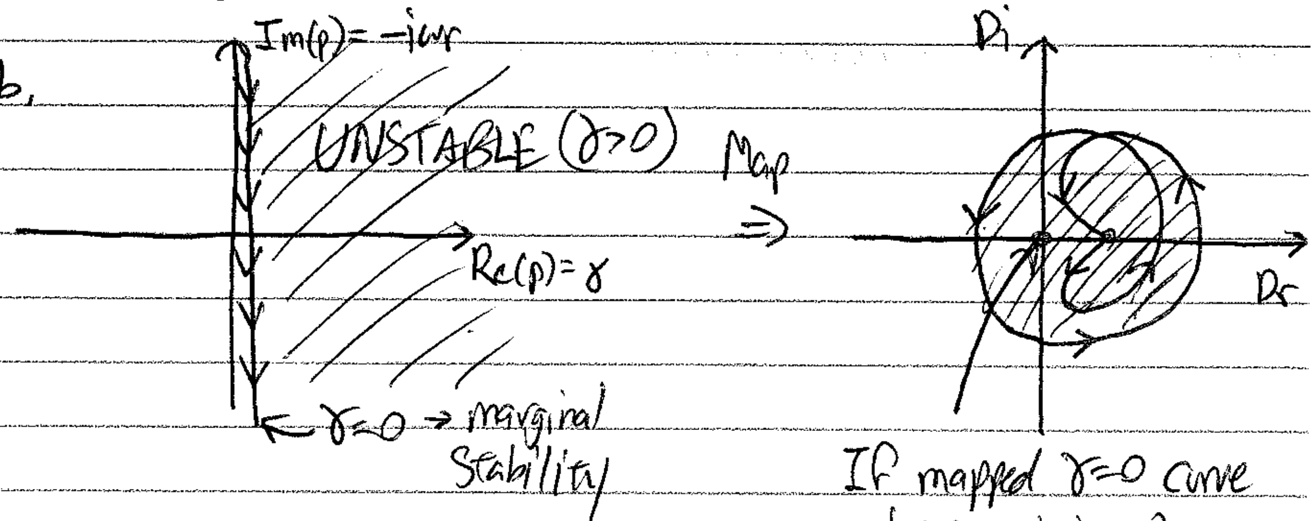
$Re(p) = \gamma$ ← growth or damping rate

$e^{-i\omega t} = e^{-i\omega_r t} e^{-i(i\gamma)t} = e^{-i\omega_r t} e^{+\gamma t}$

ii. Solution has $D_r = 0$ and $D_i = 0$

[Origin of (D_r, D_i) space]

b.



If mapped $\gamma=0$ curve encloses origin, plasma is unstable!