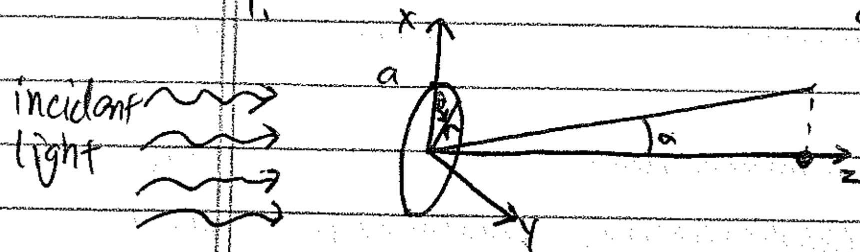
Lecture #9: Bessel Functions, Orthogonality, & Neumann Functions

I. Bessel Functions of the First Kind (Continued)

A. Ex: Fraunhofer Diffraction

1.



a. Diffraction of light of wavelength λ from a circular aperture of radius a is given by

$$\underbrace{\Phi}_{\text{wave amplitude}} \sim \int_0^a r\,dr \int_0^{2\pi} e^{\overbrace{ibr\cos\theta}^{\text{phase of radiation diffracted to } \alpha}}\, d\theta$$

where $b = \frac{2\pi}{\lambda}\sin\alpha$

2. Using $J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta}\,d\theta$, we obtain

a. $\Phi \sim 2\pi\int_0^a J_0(br)\,r\,dr$

b. Note here that imaginary part of $d\theta$ integral is zero (no need to specify Re).

3. Using $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$, we find

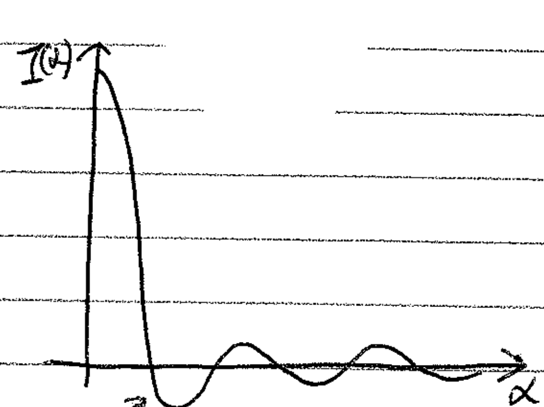
a. $\Phi \sim \frac{2\pi}{b^2}\int_0^a (br)\,J_0(br)\,d(br) = \frac{2\pi}{b^2}\int_0^a \frac{d}{d(br)}\left[(br)\,J_1(br)\right] d(br)$

$$= \frac{2\pi}{b^2}\left[br\,J_1(br)\right]_0^a = \boxed{\frac{2\pi}{b}\,a\,J_1(ba)} \qquad \text{since } J_1(0) = 0.$$

4. Thus, the intensity $I \sim \Phi^2$ of light is given by

$$I \sim \Phi^2 \sim \left[\frac{\lambda\, J_1\left[\frac{2\pi a}{\lambda}\sin\alpha\right]}{\sin\alpha}\right]^2$$

← Leads to diffractive rings

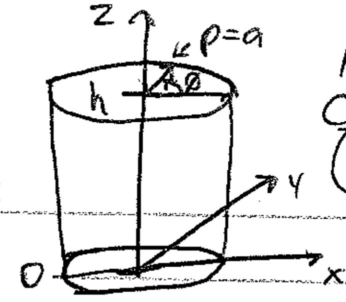


I. (Continued)

Hewes ②

Cylindrical: (ρ, ϕ, z)

B. Ex: Cylindrical Resonant Cavity



1. Consider standing electromagnetic waves in a cavity bounded by conductors at $z=0$ & $z=h$ and $\rho=a$.

2. The TM (transverse magnetic) mode has $E_z \neq 0$ and $B_z = 0$, satisfying the wave equation $$\nabla^2 E_z = +\frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

3a. For resonant standing waves, $E_z(\rho,\phi,z,t) = E_z(\rho,\phi,z)e^{i\omega t}$, so we obtain the Helmholtz Eq, $$\boxed{\nabla^2 E_z + k^2 E_z = 0} \quad \text{where} \quad \boxed{k = \frac{\omega}{c}}$$

b. Boundary Conditions: $E_\rho = E_\phi = 0$ at $z=0, z=h$ and $E_z = E_\phi = 0$ at $\rho=a$.

4. Using Separation of Variables, $E_z(\rho,\phi,z) = P_{lm}(\rho)\,\Phi_m(\phi)\,Z_l(z)$, we obtain $$\Phi_m(\phi) = e^{\pm im\phi}, \quad Z_l(z) = A\sin lz + B\cos lz$$

and $$\rho\frac{d}{d\rho}\left(\rho\frac{dP_{lm}}{d\rho}\right) + \left[(k^2 - l^2)\rho^2 - m^2\right]P_{lm} = 0 \leftarrow \text{Bessel's ODE}$$

a. Order of Bessel Function $= m$

b. Argument $n\rho = (k^2 - l^2)^{\frac{1}{2}}\rho \Rightarrow n^2 = k^2 - l^2$

c. Solution: $P_{lm}(\rho) = C\,J_m(n\rho) + \cancel{D}^{\,0}\,Y_m(n\rho)$

must be regular at $x=0$.

5. Applying Boundary Condition $E_z = 0$ at $\rho = a$.

a. $\alpha_{mj} \equiv j$th zero of J_m

b. $na = \alpha_{mj} \Rightarrow \boxed{k^2 - l^2 = \frac{\alpha_{mj}^2}{a^2}}$

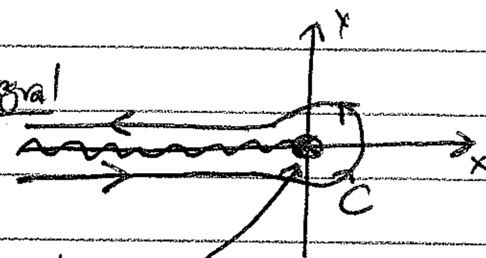
6. Apply BCs at $z=0$, $z=h$ of $\frac{\partial E_z}{\partial z} = 0$, we can obtain a final solution for resonant frequencies $$\boxed{\omega_{mjp} = c\sqrt{\frac{\alpha_{mj}^2}{a^2} + \frac{p^2\pi^2}{h^2}}}$$

I. (Continued)

C. Bessel Functions of Non-Integral Order

1. For non-integral ν, $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent

2. Integral Representation: Schlaefli Integral



a. $$J_\nu(x) = \frac{1}{2\pi i}\int_C \frac{e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}}{t^{\nu+1}}\,dt$$

b. Branch point at $t=0$, so extend branch cut to $x=-\infty$

c. Take open contour from $x=-\infty$ at $\theta=-\pi$ to $x=-\infty$ at $\theta=+\pi$.

3. A similar integral representation and open contour will be used to derive Hankel Functions (~~Sec.~~ 14.4).

II. Orthogonality

A. Bessel's ODE as a Sturm-Liouville Eigenvalue Problem

1. $$\rho^2\frac{d^2}{d\rho^2}J_\nu(k\rho) + \rho\frac{d}{d\rho}J_\nu(k\rho) + (k^2\rho^2-\nu^2)J_\nu(k\rho) = 0$$

2. Divide by ρ^2 and rearrange

a. $$-\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{\nu^2}{\rho^2}\right)J_\nu(k\rho) = k^2 J_\nu(k\rho)$$

b. $$\mathcal{L} = -\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{\nu^2}{\rho^2}\right)$$ and eigenvalue k^2

3. In the form $\mathcal{L} = p_0(x)\frac{d^2}{dx^2}+p_1(x)\frac{d}{dx}+p_2(x)$ $\quad p_0(x) = -1$, $p_1(x) = -\frac{1}{x}$

a. Operator is not self-adjoint since $p_0'(x) \neq p_1(x)$.

b. $$W(\rho) = p_0^{-1}\exp\left[\int\frac{p_1(\rho)}{p_0(\rho)}d\rho\right] = \frac{1}{(-1)}e^{+\int\frac{d\rho}{\rho}} = -e^{\ln\rho} = -\rho.$$

c. Thus we take $w(\rho) = \rho$ to make ODE self-adjoint.

II. A. (Continued)

4.a. For a Sturm Liouville Problem with eigenfunctions $f_1(x)$ & $f_2(x)$ and eigenvalues λ_1 & λ_2, the orthogonality condition is $(\lambda_1-\lambda_2)\int_a^b f_2^*(x) f_1(x) w(x)\,dx = \left[w p_0 (f_2^* f_1' - f_2^{*\prime} f_1)\right]_a^b$

b. Thus

$$\int_0^a \rho J_\nu(k\rho) J_\nu(k'\rho)\,d\rho = \frac{\left[k'a J_\nu(ka) J_\nu'(k'a) - ka J_\nu'(ka) J_\nu(k'a)\right]}{k^2-k'^2}$$

where eigenfunctions $J_\nu(k\rho)$ $J_\nu(k'\rho)$
and eigenvalues k k'

c. This integral is zero if ka & $k'a$ are zeros of J_ν!

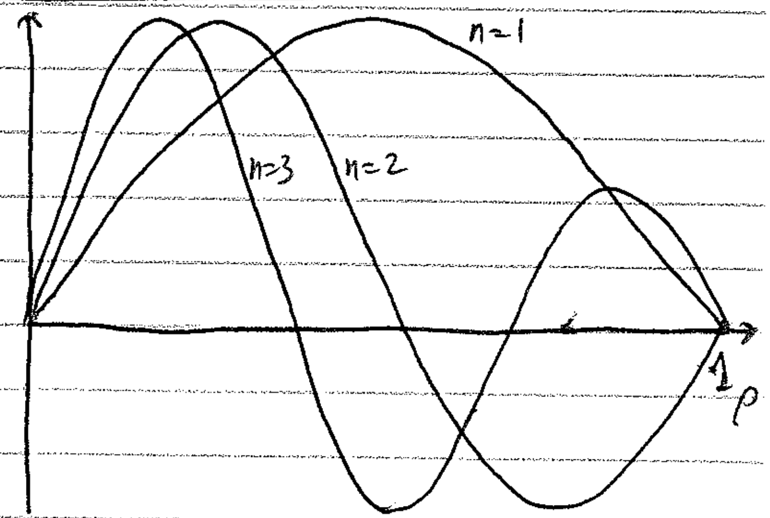
d. Thus, for $\alpha_{\nu i} \equiv$ ith zero of J_ν,

Orthogonality Formula

$$\boxed{\int_0^a \rho J_\nu\left(\alpha_{\nu i}\frac{\rho}{a}\right) J_\nu\left(\alpha_{\nu j}\frac{\rho}{a}\right) d\rho = 0 \quad \text{if } i \neq j}$$

e. NOTE: All members of orthogonal set have same value of ν.

f. $J_1(\alpha_{1n}\rho)$



B. Normalization

1. To derive normalization, take $\lim_{k'\to k}$ (using L'Hopital's Rule for RHS) and chose $ka = \alpha_{\nu i}$,

$$\int_0^a \rho \left[J_\nu\left(\alpha_{\nu i}\frac{\rho}{a}\right)\right]^2 d\rho = \frac{a^2}{2}\left[J_\nu'(\alpha_{\nu i})\right]^2$$

II. B. (Continued)

2. To simplify, use $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

a. Here $x = \alpha_{\nu i}$, So $J_\nu(\alpha_{\nu i}) = 0$, but $J_{\nu+1}(\alpha_{\nu i}) \neq 0$, so

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -n x^{-n-1} \cancel{J_n(x)}^{\,0\,(zero)} + x^{-n} J_n'(x) = -x^{-n} J_{n+1}(x)$$

$$\Rightarrow \quad J_\nu'(\alpha_{\nu i}) = -J_{\nu+1}(\alpha_{\nu i})$$

3. Thus, $\boxed{\int_0^a \rho \left[J_\nu\left(\alpha_{\nu i} \frac{\rho}{a}\right)\right]^2 d\rho = \frac{a^2}{2}\left[J_{\nu+1}(\alpha_{\nu i})\right]^2}$

C. Expansion of Functions in Series of Bessel Functions

1. For fixed ν, the set of Bessel functions $J_\nu\left(\alpha_{\nu j} \frac{\rho}{a}\right)$ is complete.

2. Thus $\boxed{f(\rho) = \sum_{j=1}^{\infty} c_{\nu j} J_\nu\left(\alpha_{\nu j} \frac{\rho}{a}\right)}$, $0 \le \rho \le a$ $\quad \nu > -1$

with coefficients $\boxed{c_{\nu j} = \frac{2}{a^2\left[J_{\nu+1}(\alpha_{\nu j})\right]^2} \int_0^a f(\rho) J_\nu\left(\alpha_{\nu j} \frac{\rho}{a}\right) \rho\, d\rho}$

III. Neumann Functions (Bessel Functions of the Second Kind)

A. Definition

1. Neumann functions: $\boxed{Y_\nu(x) = \frac{\cos \nu\pi \, J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}}$

a. For non-integral ν, just linear combination of two linearly independent solutions $J_{-\nu}(x)$ and $J_\nu(x)$

b. But, for $\nu \to n$ (integer), this definition is indeterminate.

c. Using L'Hôpital's Rule with $Y_n(x) = \lim_{\nu \to n} Y_\nu(x)$ yields

$$\boxed{\begin{aligned} Y_n(x) = &\frac{2}{\pi} J_n(x) \ln\left(\frac{x}{2}\right) - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-n} \\ &- \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left[\psi(k+1) + \psi(n+k+1)\right] \left(\frac{x}{2}\right)^{2k+n} \end{aligned}}$$

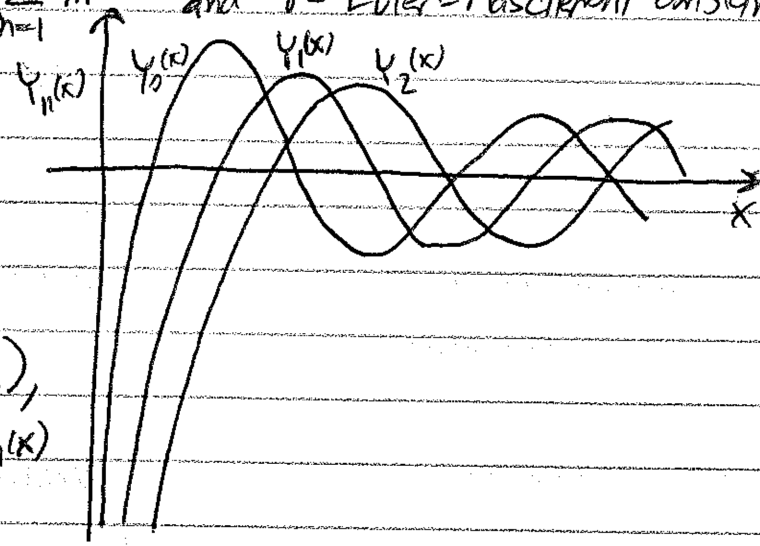
where $\psi(x)$ is digamma function

III A. (Continued)

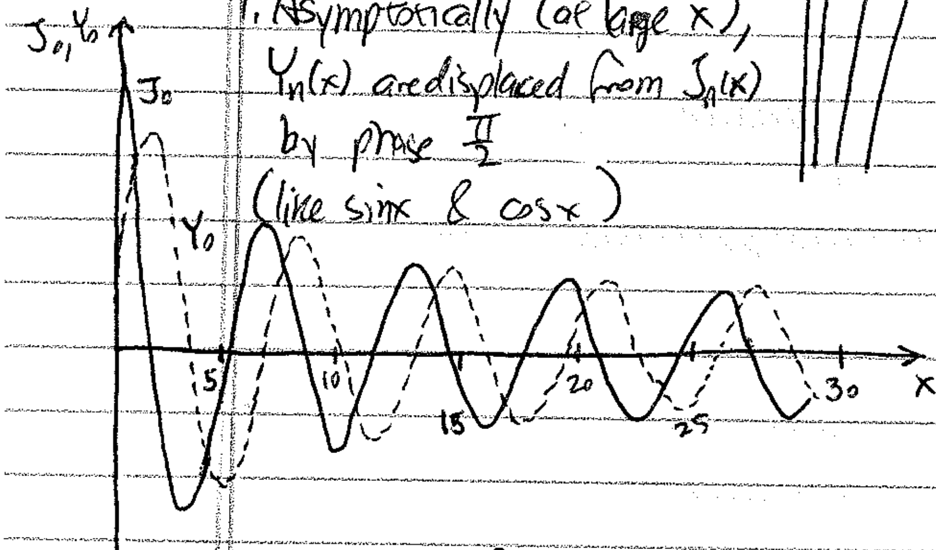
2. For $Y_0(x) = \frac{2}{\pi} J_0(x)\left[\gamma + \ln\left(\frac{x}{2}\right)\right] - \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{k!\,k!} H_k \left(\frac{x}{2}\right)^{2k}$

where $H_k = \sum_{m=1}^{k} m^{-1}$ and $\gamma \equiv$ Euler-Mascheroni constant.

3. Neumann functions are irregular at $x=0$.



4. Asymptotically (at large x), $Y_n(x)$ are displaced from $J_n(x)$ by phase $\frac{\pi}{2}$ (like $\sin x$ & $\cos x$)



5. Integral Representation

$$Y_0(x) = -\frac{2}{\pi}\int_0^{\infty} \cos(x\cosh t)\,dt = -\frac{2}{\pi}\int_1^{\infty} \frac{\cos(xt)}{(t^2-1)^{1/2}}\,dt \qquad x>0$$

6. Recurrence Relations

a. Satisfies the same recurrence and derivative relations as $J_\nu(x)$:

$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$$

$$Y_{n-1}(x) - Y_{n-1}(x) = 2\,Y_n'(x)$$

III. (Continued)

B. Wronskian Formulas

1. Recall for an ODE $p(x)\,y'' + q(x)\,y' + r(x)\,y = 0$ in self-adjoint form (such that $p' = q$), the Wronskian formula connects solutions u & v: $u(x)v'(x) - u'(x)v(x) = \dfrac{A}{p(x)}$

2a. Bessel's ODE in self-adjoint form: $x y'' + y' + \left(x - \frac{\nu^2}{x}\right)y = 0$, $\rightarrow p(x) = x$

b. We obtain for non-integer ν
$$\boxed{J_\nu J_{-\nu}' - J_\nu' J_{-\nu} = \frac{A_\nu}{x}}$$

c. Note that A_ν is a constant, independent of x, but may depend on ν. Thus, we can determine A_ν at any convenient point, such as $x=0$

d. In the limit $x \to 0$, we can use just the first term of power series expansion

$$\lim_{x\to 0} J_\nu = \frac{1}{\Gamma(1+\nu)}\left(\frac{x}{2}\right)^\nu \Rightarrow \lim_{x\to 0} J_\nu' = \frac{\nu}{2\Gamma(1+\nu)}\left(\frac{x}{2}\right)^{\nu-1}$$

$$\lim_{x\to 0} J_{-\nu} = \frac{1}{\Gamma(1-\nu)}\left(\frac{x}{2}\right)^{-\nu} \Rightarrow \lim_{x\to 0} J_{-\nu}' = \frac{-\nu}{2\Gamma(1-\nu)}\left(\frac{x}{2}\right)^{-\nu-1}$$

e. Thus

$$J_\nu J_{-\nu}' - J_\nu' J_{-\nu} = \frac{-\left(\frac{x}{2}\right)^\nu\left(\frac{x}{2}\right)^{-\nu-1}}{2\Gamma(1+\nu)\Gamma(1-\nu)} - \frac{\left(\frac{x}{2}\right)^{\nu-1}\left(\frac{x}{2}\right)^{-\nu}}{2\Gamma(1+\nu)\Gamma(1-\nu)}$$

$$= \frac{-2\nu}{x\Gamma(1+\nu)\Gamma(1-\nu)}$$

f. NOTE: Using the reflection formula for the Γ function (eq. 13.23),
$\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin z\pi}$, we obtain

$$\frac{A_\nu}{x} = -\frac{2\sin\nu\pi}{\pi x}$$

g. Thus $\boxed{A_\nu = \dfrac{-2\sin\nu\pi}{\pi}}$ Vanishes for $\nu = n$ (integer), showing that J_n and J_{-n} are not linearly independent.

3. Recurrence relations can be used to derive a number of alternate forms,

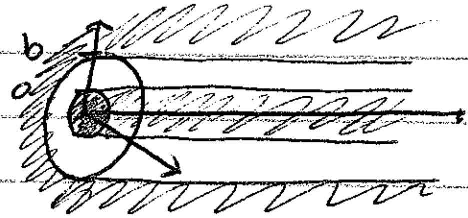
$$J_\nu J_{-\nu+1} + J_{-\nu} J_{\nu-1} = \frac{2\sin\nu\pi}{\pi x}$$

$$J_\nu Y_\nu' - J_\nu' Y_\nu = \frac{2}{\pi x}, \text{ etc.}$$

III. (Continued)

C. Use of Neumann Functions

1. Second, independent solution of Bessel's Equation
2. Needed often for problems that don't require regularity at $x=0$ (EM waves in coaxial geometry, quantum mechanical scattering)
3. Lead to Hankel functions (next section).



D. Ex: Coaxial Wave Guides

1. Consider EM wave propagation between two concentric, conducting cylinders at $\rho=a$, $\rho=b$.
2. Same equations as Example I. B. in this lecture, but different BC's
   a. Separation of Variables $E_z(\rho,\phi,z) = P(\rho)\Phi(\phi)Z(z)$
   b. We want traveling waves, so $Z(z) = e^{i(lz-\omega t)}$
3. Thus, general solution

$$E_z(\rho,\phi,z) = \left[c_{mn}J_m(\gamma_{mn}\rho) + d_{mn}Y(\gamma_{mn}\rho)\right]e^{\pm im\phi}e^{i(lz-\omega t)}$$

where $\frac{\omega^2}{c^2} = \gamma_{mn}{}^2 + l^2$ and l may have any value (no BC in z).

4. TM traveling wave must have $E_z = 0$ at $\rho=a$ & $\rho=b$, so

$$\boxed{\begin{aligned} c_{mn}J_m(\gamma_{mn}a) + d_{mn}Y_m(\gamma_{mn}a) &= 0 \\ c_{mn}J_m(\gamma_{mn}b) + d_{mn}Y_m(\gamma_{mn}b) &= 0 \end{aligned}}$$

   a. Transcendental equations, must be solved numerically for c_{mn} and d_{mn} given a frequency ω & l (which determines γ_{mn}).

5. Cutoff Frequency: Taking limit $l\to 0$, minimum $\omega = c\gamma_{mn}$.
   a. TM waves can only propagate above cutoff frequency,

$$\boxed{\omega > \omega_{cutoff} = c\gamma_{mn}}$$