PHYS:7729 Homework #2

Reading:	Required:	Read Gurnett & Bhattacharjee (GB), Chapter 6, Sec 6.4–6.6 (p.202–217)
	Required:	Read Gurnett & Bhattacharjee (GB), Chapter 7, Sec 7.1–7.2 (p.221–239)
	Optional:	Read Boyd & Sanderson (BS), Chapter 4, Sections 4.3–4.4 (p.82–107)

Due at 5:00pm, Friday, February 19, 2021.

1. Fluid Electron Waves
   Assuming that the ions are stationary in a homogeneous, unmagnetized plasma and that the electrons respond to an applied electrostatic wave of the form
   $$\phi(x,t) = \phi_1 e^{i(kx-\omega t)},$$
   use the electron equations from the Two Fluid Equations (neglecting the drag term)
   $$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{U}_e) = 0$$
   $$dm_e n_e \left(\frac{\partial \mathbf{U}_e}{\partial t} + \mathbf{U}_e \cdot \nabla \mathbf{U}_e \right) = -\nabla p_e - e n_e \left(\mathbf{E} + \mathbf{U}_e \times \mathbf{B} \right)$$
   $$\frac{d}{dt} \left(\frac{p_e}{n_e^\gamma} \right) = 0$$
   along with Poisson's Equation
   $$\nabla \cdot \mathbf{E} = \frac{\sum_s n_s q_s}{\epsilon_0}$$
   to derive the dispersion relation $\omega = \omega(k)$. Hint: Use $\mathbf{E} = -\nabla \phi$ and you may use the expression $p_e = n_e k T_e$.

   (a) Treating the perturbation as small, write down the linearized set of equations. Hint: Don't forget to use the the property of quasineutrality.

   (b) Solve for the dispersion relation $\omega = \omega(k)$ in terms of the electron plasma frequency ω_{pe} and the electron thermal velocity v_{te}.

   (c) What is the appropriate adiabatic index γ for these waves if they are very fast (and thus adiabatic) and strictly one-dimensional?

   (d) What is the group velocity for these waves?

2. From the general form for linearized Ideal MHD,
   $$\omega^2 \mathbf{U}_1 = (c_s^2 + v_A^2)(\mathbf{k} \cdot \mathbf{U}_1)\mathbf{k} - v_A^2 (\hat{\mathbf{b}} \cdot \mathbf{U}_1)(\hat{\mathbf{b}} \cdot \mathbf{k})\mathbf{k} - v_A^2 (\hat{\mathbf{b}} \cdot \mathbf{k})(\mathbf{k} \cdot \mathbf{U}_1)\hat{\mathbf{b}} + v_A^2 (\hat{\mathbf{b}} \cdot \mathbf{k})^2 \mathbf{U}_1,$$
   solve for the MHD Dispersion Relation in the form $D(\omega, \mathbf{k}) = 0$ (this is the determinant set equal to zero).

3. One-Dimensional Solar Wind Model

Consider a simplified, steady-state (constant in time) model of the solar wind near the equatorial plane in which all quantities depend only on the spherical radius r. Assume the radial component of the solar wind velocity is a given function $v_r(r)$. Hint: Use spherical coordinates, and take $B_\theta = 0$.

(a) Use the MHD continuity equation to derive the mass density ρ as a function of radius r in terms of the solar radius $R_\odot$, the density at the solar surface $\rho_\odot$, and the radial velocity at the solar surface $v_r(R_\odot)$.

(b) Use divergence free condition on the magnetic field to determine the radial component of the magnetic field B_r as a function of radius r in terms of the solar radius $R_\odot$ and the radial field at the solar surface $B_r(R_\odot)$. NOTE: This is an expression of the consecration of magnetic flux.

(c) Use the azimuthal component of the Ideal MHD induction equation to derive an expression from B_ϕ as a function of radius r in terms of the angular rotational velocity at the solar surface $\Omega_\odot$, the radial component of the magnetic field B_r, radial solar wind velocity v_r, and the azimuthal solar wind velocity v_ϕ. You may assume that the magnetic field at the solar surface is purely radial $B_\phi(R_\odot) = 0$. Hint: Use your results from part (b) to simplify the expression and express the v_ϕ at the solar surface as a function of $\Omega_\odot$.

4. A force-free equilibrium, in which the thermal pressure gradient forces are negligible and the magnetic force $\mathbf{j} \times \mathbf{B} = 0$, can be satisfied if the current is parallel to the magnetic field. Since the current density in MHD is given by Ampere's Law $\mu_0 \mathbf{j} = \nabla \times \mathbf{B}$, a magnetic field that satisfies the relation

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}$$

will yield a force-free equilibrium, where α is a scalar that may or may not be constant.

(a) In cylindrical coordinates, show that a magnetic field of the form

$$\mathbf{B} = [B_\rho, B_\phi, B_z] = \left[0, \frac{B_0 k \rho}{1 + k^2 \rho^2}, \frac{B_0}{1 + k^2 \rho^2}\right]$$

where k and B_0 are constants, yields a force-free equilibrium. Solve for the form of α.

(b) Compute the form of the current density along the axial direction, j_z, due to this magnetic field.