Hawes 1

PHYS: 7729 Plasma Physics II

Final Exam Review:

I. Magnetohydrodynamics (MHD)

A. MHD Approximation

1. Strong Collisions: $\lambda_m \ll L,\ \tau \gg \frac{1}{\nu_{ei}}$
2. Non-relativistic: $V_0^2/c^2 \ll 1$
3. Magnetized: $r_{Li} \ll L$

$\Big\}$ Quasi-neutrality $\sum_s n_s q_s \simeq 0$

B. Derivation from Two-Fluid Equations

1. Drop $\underset{\approx}{\Pi}$, viscous terms from $\underset{\approx}{P}$ tensor (Collisional)
2. Isotropic pressure (Collisional)
3. Drop Displacement Current (Non-relativistic)
4. Drop Hall term in Ohm's Law (Magnetized $r_{Li} \gg L$)
5. Adiabatic Equation of State

C. MHD Equations

1. Continuity $\quad \frac{\partial \rho}{\partial t} + \underset{\sim}{U}\cdot\nabla\rho = -\rho\nabla\cdot\underset{\sim}{U}$

2. Momentum $\quad \rho\frac{\partial U}{\partial t} + \rho\underset{\sim}{U}\cdot\nabla\rho = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{(\underset{\sim}{B}\cdot\nabla)\underset{\sim}{B}}{\mu_0}$

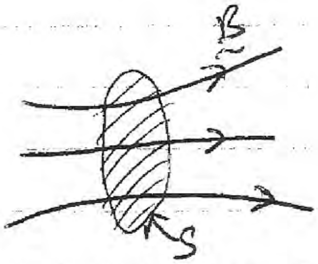
3. Induction $\quad \frac{\partial B}{\partial t} = \nabla\times(\underset{\sim}{U}\times\underset{\sim}{B}) + \frac{\eta}{\mu_0}\nabla^2\underset{\sim}{B}$

4. Adiabatic Equation of State $\quad \frac{\partial p}{\partial t} + \underset{\sim}{U}\cdot\nabla p = -\gamma p\nabla\cdot\underset{\sim}{U}$

D. Frozen-In Flux

1. Magnetic Reynolds Number: $Re_M \equiv \frac{\mu_0 L V_0}{\eta}$
2. Limits: a) $Re_M \gg 1 \Rightarrow$ Ideal MHD
   b) $Re_M \ll 1 \Rightarrow$ Diffusive

3. Frozen-In Flux Theorem: (Ideal Limit)



a. Magnetic Flux $\Phi_B = \int_S \underset{\sim}{B} \cdot d\underset{\sim}{A}$

b. Magnetic Field lines are frozen to the flow

4. Clebsch Coordinates:

a. $\underset{\sim}{B} = \nabla\alpha \times \nabla\beta$

b. $\left.\begin{array}{l}\alpha(\underset{\sim}{x}) = \text{const} \\ \beta(\underset{\sim}{x}) = \text{const}\end{array}\right\}$ Defines field lines $\underset{\sim}{B}(\underset{\sim}{x})$

5. Applications of Frozen-In Flux: $\Phi_B = $ constant as dA changes!

E. MHD Linear Dispersion Relation

1. Know the steps to compute the linear dispersion relation!

a. Linearize the equations: $\rho = \underset{\text{uniform, steady}}{\rho_0} + \epsilon \rho_1(\underset{\sim}{x})$, $\underset{\sim}{U} = \cancel{\underset{\sim}{U}_0}^{\,0} + \epsilon \underset{\sim}{U}_1(\underset{\sim}{x})$, etc.

b. Fourier transform (plane-wave solution)

$$e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)} \Rightarrow \quad \nabla \Rightarrow i\underset{\sim}{k}$$
$$\frac{\partial}{\partial t} \Rightarrow -i\omega$$

I. (Continued)

E. (Continued)

c. Eliminate ρ_1, $\underset{\sim}{B}_1$, p_1 in terms of $\underset{\sim}{U}_1$

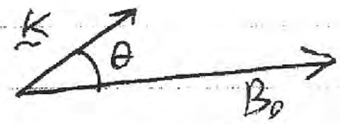
d. Collect to obtain matrix equation

$$\underset{\approx}{M} \cdot \underset{\sim}{U} = 0$$

e. Set Determinant $|\underset{\approx}{M}| = 0 \Rightarrow$ Yields $D(\omega, \underset{\sim}{k}) = 0$

f. Solutions for ω (in terms of $\underset{\sim}{k}$ and other parameters) yield the linear wave modes!

F. MHD Wave Modes



1. $\underset{\sim}{k} = k_\perp \hat{\underset{\sim}{x}} + k_\parallel \hat{\underset{\sim}{z}} = k \sin\theta \hat{\underset{\sim}{x}} + k\cos\theta \hat{\underset{\sim}{z}}$, $\underset{\sim}{B} = B_0 \hat{\underset{\sim}{z}}$

2. General Dispersion Relation

$$\underbrace{(\omega^2 - k_\parallel^2 v_A^2)}_{\text{Alfven Modes, incompressible}} \underbrace{\left[\omega^4 - \omega^2 k^2 (c_s^2 + v_A^2) + k_\parallel^2 k^2 c_s^2 v_A^2\right]}_{\text{Fast and Slow Magnetosonic Waves, compressible}} = 0$$

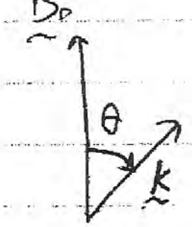
$$\omega = \pm k_\parallel v_A \qquad \frac{\omega^2}{k^2} = \frac{1}{2}(c_s^2 + v_A^2) \pm \frac{1}{2}\sqrt{(c_s^2 + v_A^2) - 4 c_s^2 v_A^2 \cos^2\theta}$$

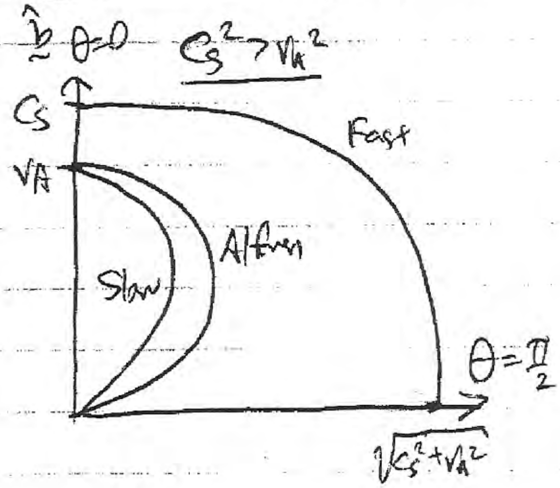
3. Sound speed $\quad c_s^2 = \frac{\gamma p_0}{\rho_0}$ $\qquad$ Alfven Speed $\quad v_A^2 = \frac{B_0^2}{\mu_0 \rho_0}$

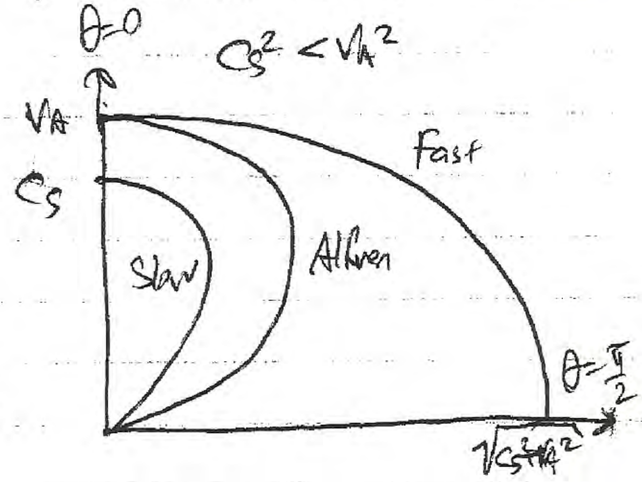
4. a. Fast: Thermal & Magnetic Pressure add

Slow: Thermal & Magnetic Pressure oppose

5. Polar Plots: $v_p = \frac{\omega}{k}$ = phase speed







G. Conserved Energy

1. $E = \int d^3x \left(\frac{1}{2}\rho U^2 + \frac{P}{\gamma-1} + \frac{B^2}{2\mu_0} \right)$

H. Eigenfunctions of MHD Waves

1. $$\begin{pmatrix} (-) & 0 & (-) \\ 0 & (-) & 0 \\ (-) & 0 & (-) \end{pmatrix} \begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix} = 0$$

2. Take $U_y = 0$, $U_x = U_0$

3. Solve for all other quantities in terms of U_x using the linearized, Fourier-transforms equations

$\Rightarrow \rho_1, \underset{\sim}{U_1}, \underset{\sim}{B_1}, P_1$ in terms of $U_x = U_0$!

4. Plug in ω for particular wave mode.

II. MHD Equilibrium

1. Force-Balanced Equilibrium: $\nabla p = \underset{\sim}{j} \times \underset{\sim}{B}$

2. Hopf's Thm: Torus is simplest surface such that $\underset{\sim}{B} \cdot \nabla p = 0$.

3. Force-Free Equilibrium: $\underset{\sim}{j} \times \underset{\sim}{B} = 0$

   a $\Rightarrow$ $\nabla \times \underset{\sim}{B} = \alpha \underset{\sim}{B}$ $\Rightarrow$ $\underset{\sim}{B} \cdot \nabla \alpha = 0$
   surfaces of constant α

4. Cylindrical Equilibria: $\nabla(\underset{P_T}{p} + \underset{P_B}{\frac{B^2}{2\mu_0}}) = \underset{\text{Mag tension}}{\frac{(\underset{\sim}{B} \cdot \nabla)\underset{\sim}{B}}{\mu_0}}$

   a. $\underset{\sim}{B} = B_\phi(r)\hat{\phi} + B_z(r)\hat{z}$ (Ex. Flux rope, RFP)

   b. $\frac{d}{dr}\left(p + \frac{B_\phi^2}{2\mu_0} + \frac{B_z^2}{2\mu_0}\right) = -\frac{B_\phi^2}{\mu_0 r}$

   c. $B_z = 0 \Rightarrow$ Z-pinch
   $B_\phi = 0 \Rightarrow$ Theta-pinch

5. Toroidal Equilibria: (Not on Exam)

   a. Magnetic Flux Coordinates $[\psi(r,z), F(\psi)]$

   b. Grad-Shafranov Equation
   $\Rightarrow$ Magnetostatic Equilibria

III. Multiple Timescale Methods

A. General Approach

1. Two timescales: t, $\tau = \epsilon^2 t$

2. $\frac{d}{dt} = \frac{\partial}{\partial t} + \epsilon^2 \frac{\partial}{\partial \tau}$

3. Expand variables: $x = \cancel{x_0}^{\,0} + \epsilon x_1 + \epsilon^2 x_2 + \ldots$

4. Substitute into equation and multiply out to get all terms

5. Collect terms order by order to obtain
   $O(1)$
   $O(\epsilon)$ equations
   $O(\epsilon^2)$

6. Solve Equations order by order for x_1, x_2, ...
   $\Rightarrow$ set $\epsilon = 1$!

7. NOTE: Every problem is different. I will try to walk you through any problems a give you (Prob#1, Hwk3)

III. (Continued)

B. Polarization Drift

1. Slowly varying electric field, $\underset{\sim}{E}(\tau)$ with constant $\underset{\sim}{B} = B_0 \hat{z}$.

2. $$\underset{\sim}{v} = \underbrace{v_\perp \left(\cos(\omega_c t + \phi)\hat{x} - \sin(\omega_c t + \phi)\hat{y} \right)}_{\substack{O(1)\text{: Larmor Orbit}}} + \underbrace{\frac{\underset{\sim}{E}(\tau) \times \underset{\sim}{B}}{B_0^2}}_{\substack{O(\epsilon) \\ E\times B \text{ drift}}} + \underbrace{\frac{1}{\omega_c B_0} \frac{d\underset{\sim}{E}}{dt}}_{\substack{O(\epsilon^2) \\ \text{Polarization} \\ \text{Drift, } \underset{\sim}{v}_p}}$$

3. $$\underset{\sim}{v}_p = \frac{1}{\omega_c B} \frac{d\underset{\sim}{E}}{dt} = \frac{m}{qB^2} \frac{d\underset{\sim}{E}}{dt}$$

C. Ponderomotive Force

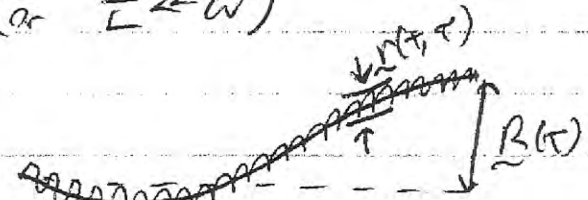
1. Particle motion in rapidly varying electromagnetic field

$$\underset{\sim}{E}(\underset{\sim}{x}, t, \tau) = \underset{\sim}{E}_0(\underset{\sim}{x}, \tau) \cos(\omega t - \underset{\sim}{k} \cdot \underset{\sim}{x})$$

fast slow

2. Ordering: $\epsilon = \frac{V}{L\omega} \ll 1$ (or $\frac{V}{L} \ll \omega$)

3. $$\underset{\sim}{x} = \underset{\sim}{R}(\tau) + \epsilon\, \underset{\sim}{r}(t, \tau)$$



$\underset{\sim}{R}$: oscillation center

4. $$\underset{\sim}{F}_{pond} = m \frac{d\underset{\sim}{u}}{dt} = \frac{-q^2}{4m\omega^2} \nabla |\underset{\sim}{E}_0|^2$$

$\underset{\sim}{E}_0(\underset{\sim}{x})$

NOTE: Drop $\cos(\omega t - \underset{\sim}{k} \cdot \underset{\sim}{x})$ in this part!

5. $$\underset{\sim}{F}_{pond} = -\nabla \Phi_{pond} \qquad \Phi_{pond} = \frac{q^2}{4m\omega^2} |\underset{\sim}{E}_0|^2$$

6. Energy Picture: $\mathcal{E}_{osc} = \frac{1}{2} m u^2 + \Phi_{pond} = \text{constant!}$

Hawes ⑧

IV MHD Stability

A. Will we be tested on the Final Exam

V. Kinetic Theory: Linear Waves and Landau Damping

A. Equations:

1. Boltzmann Eq: $\frac{\partial f_s}{\partial t} + \underset{\sim}{v} \cdot \nabla f_s + \frac{q_s}{m_s}(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} = \left(\frac{\partial f_s}{\partial t}\right)_{coll}$

a. Vlasov Equation when $\left(\frac{\partial f_s}{\partial t}\right)_{coll} = 0$.

b. $f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)$: 6D (3D-3V) Distribution Function)

2. Maxwell's Equations:

$$\nabla \cdot \underset{\sim}{E} = \frac{\rho_q}{\epsilon_0} \qquad \rho_q = \sum_s \int d^3\underset{\sim}{v}\, q_s f_s$$

$$\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}$$

$$\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{j} + \mu_0 \epsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t} \qquad \underset{\sim}{j} = \sum_s \int d^3\underset{\sim}{v}\, q_s \underset{\sim}{v} f_s$$

$$\nabla \cdot \underset{\sim}{B} = 0$$

B. Equilibrium and Moments:

1. Maxwellian Equilibrium: $f_{s_m}(\underset{\sim}{x}, \underset{\sim}{v}, t) = \frac{n_s(\underset{\sim}{x}, t)}{\pi^{3/2} v_{ts}^{3/2}} e^{-\frac{m_s |\underset{\sim}{v} - U_s(\underset{\sim}{x}, t)|^2}{2 T_s(\underset{\sim}{x}, t)}}$

a. $v_{ts}^2 = \frac{2 T_s}{m_s}$ b. Uniform: $f_{s_m}(\underset{\sim}{v}) = \frac{n_{s0}}{\pi^{3/2} v_{ts}^{3/2}} e^{-\frac{v^2}{v_{ts}^2}}$

2. Moments: a. $n_{s_0} = \int d^3v \; f_{s_m}(\underset{\sim}{v})$

b. $\frac{3}{2} n_{s_0} T_{s_0} = \int d^3\underset{\sim}{v} \; \frac{1}{2} m_s v^2 \; f_{s_m}(\underset{\sim}{v})$

3. Reduced Distribution Function:

a. $F_s(v_z) \equiv \frac{1}{n_{s_0}} \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \; f_s(\underset{\sim}{v})$

b. $\int_{-\infty}^{\infty} dv_z \; F_s(v_z) = 1$

C. Electrostatic Waves in an Unmagnetized Plasma

1. $\underset{\sim}{E} = -\nabla \phi$

2. Linearized: a. $\frac{\partial f_{s_1}}{\partial t} + \underset{\sim}{v} \cdot \nabla f_{s_1} - \frac{q_s}{m_s} \nabla \phi \cdot \frac{\partial f_{s_0}}{\partial \underset{\sim}{v}} = 0$

b. $-\nabla^2 \phi_1 = \frac{1}{\epsilon_0} \sum_s \int d^3\underset{\sim}{v} \; q_s f_{s_1}$

3. Fourier Transform Solution: $\frac{\partial}{\partial t} \to -i\omega, \; \nabla \to i\underset{\sim}{k}$

a. $D(\underset{\sim}{k}, \omega) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\partial F_{s_0}/\partial v_z}{(v_z - \frac{\omega}{k})} = 0$

b. Only valid when $F_{s_0}(v_z = \frac{\omega}{k}) = 0$ and $\left. \frac{\partial F_{s_0}}{\partial v_z} \right|_{v_z = \frac{\omega}{k}} = 0$

c. Integrate by parts: $D(\underset{\sim}{k}, \omega) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{F_{s_0}}{(v_z - \frac{\omega}{k})^2} = 0$

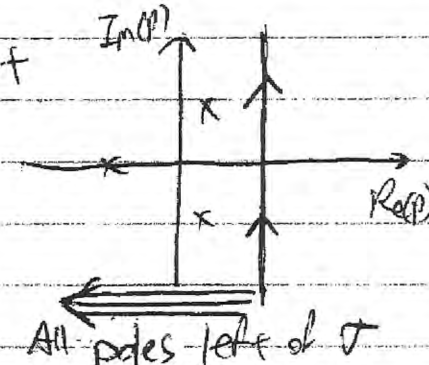
V. (Continued)

D. Laplace Transforms and Contour Integration

1. Laplace Transforms:

a. $\tilde{f}(p) = \int_0^\infty dt\, f(t)\, e^{-pt} \qquad Re(p) > 0$

b. Inverse: $f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} dp\, \tilde{f}(p) e^{pt}$



c. $\tilde{f}'(p) = p\tilde{f}(p) - f(t=0)$

2. Contour Integration

a. Analytic Function: Derivative of $f(z)$ everywhere

b. Deformation of paths

c. Cauchy Integral Formula: $f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z - z_0}$

d. Residue Theorem: $\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} [f(z)]$

3. Know how to solve a system of equations using a Laplace–Fourier transform approach!

time   space

V. (Continued)

E. Laplace-Fourier Solution for Electrostatic Plasma Waves

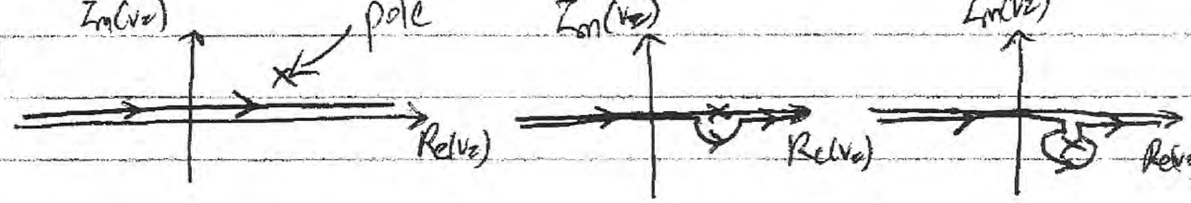
1. Solution: $\tilde{\phi}_1(\underline{k},p) = \dfrac{N(\underline{k},p)}{D(\underline{k},p)}$

a. $N(\underline{k},p) = -i \sum_s \frac{q_s n_0}{\epsilon_0 k^3} \int_{-\infty}^{\infty} dv_z \frac{F_s(0)}{v_z - \frac{ip}{k}}$

b. Dispersion Relation: $D(\underline{k},p) = 0$

$$D(\underline{k},p) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\partial f_{s0}/\partial v_z}{v_z - \frac{ip}{k}} = 0$$

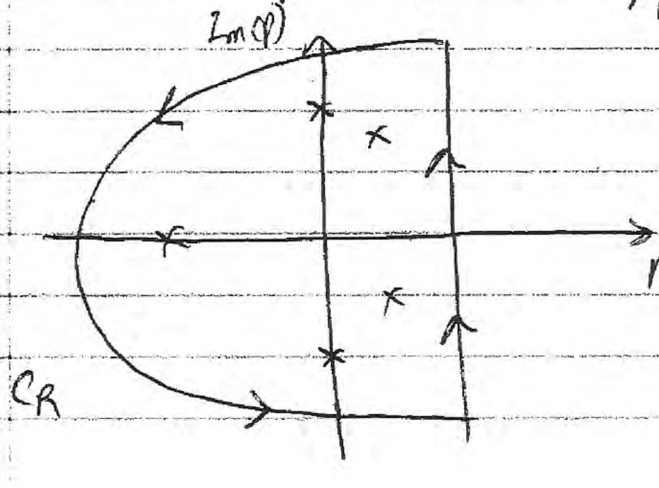
c. Landau Contour for velocity space integral:

i) 

ii) Contour always goes under the pole.

iii) Achieves analytic continuation of $D(\underline{k},p)$ to $Re(p)<0$!

d. Inverse Laplace Transform: $\hat{\phi}_1(k,t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} dp\, \tilde{\phi}_1(\underline{k},p) e^{pt}$



e. Close Contour (C_R)

f. Evaluate using Residue Theorem

V.E. (Continued)

2. Know how to solve for $D(\underset{\sim}{k}, p) = 0$ for a given choice of $F_0(v_z)$!

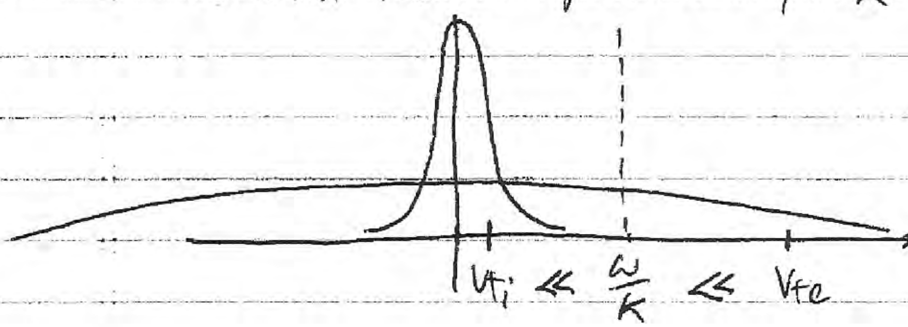
a. Examples: Cauchy Distribution: $F_{0c}(v_z) = \frac{c}{\pi} \frac{1}{c^2 + v_z^2}$

b. For Maxwellian Distribution ⇒ Plasma Dispersion Function! (see below)

F. Weak Growth Rate Approximation

1. Can be used anytime $|\gamma| \ll |\omega|$

a. When resonant phase velocity $\frac{\omega}{k}$ is not near v_ts!



i) $\frac{\omega}{k} \gg v_{ti}$ ⇒ weak ion damping!

ii) $\frac{\omega}{k} \ll v_{te}$ ⇒ weak electron damping!

2. a. Solve for Complex function $D(\underset{\sim}{k}, p) = 0$

$D(\underset{\sim}{k}, -i\omega) = 0$ ← $\omega = \omega_r + i\gamma$

b. Separate real and imaginary parts: $D(\underset{\sim}{k}, -i\omega) = D_r(\underset{\sim}{k}, -i\omega) + i D_i(\underset{\sim}{k}, -i\omega)$

c. Set $\gamma = 0$ → $\omega = \omega_r + \cancel{i\gamma}^{\,0} = \omega_r$!

V. F. (Continued) Howes (13)

3. Weak Growth Rate Approximation:

a. $\boxed{D_r(\underline{k}, -i\omega_r) = 0}$ ⇒ solve for ω_r!

b. $\boxed{\gamma = \frac{-D_i(\underline{k}, -i\omega_r)}{\partial D_r(\underline{k}, -i\omega_r)/\partial \omega_r}}$ ⇒ solve for γ!

4. Application to Electrostatic Waves:

a. Plemelj Relation: i) $\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} dx \frac{f(x)}{x-(x_0 \pm i\epsilon)} = P\int_{-\infty}^{\infty} dx \frac{f(x)}{x-x_0} \pm i\pi f(x_0)$

ii) Principle Value: $P\int_{-\infty}^{\infty} dx \frac{g(x)}{x-x_0} = \lim_{\delta \to 0} \left[\int_{-\infty}^{x_0-\delta} dx \frac{g(x)}{x-x_0} + \int_{x_0+\delta}^{\infty} dx \frac{g(x)}{x-x_0} \right]$

b. For equilibrium $F_{s0}(v_z)$,

$$\gamma = \pi \frac{k}{|k|} \frac{1}{\frac{\partial D_r(k,\omega_r)}{\partial \omega_r}} \sum_s \frac{\omega_{ps}^2}{k^2} \left. \frac{\partial f_{0s}}{\partial v_z} \right|_{v_z = \frac{\omega}{k}}$$

G. Plasma Dispersion Function: For Maxwellian equilibrium,

1. $Z(\xi_s) = \int_C \frac{dz}{\pi^{\frac{1}{2}}} \frac{e^{-z^2}}{z-\xi_s}$ $\quad \xi_s = \frac{ip}{kv_{ts}} = \frac{\omega}{kv_{ts}} - i\frac{\gamma}{kv_{ts}}$

2. Limits:

a. $|\xi| \gg 1$: $Z(\xi) = i\pi^{\frac{1}{2}} e^{-\xi^2} - \frac{1}{\xi} - \frac{1}{2\xi^3} - \frac{3}{4\xi^5} - \dots$

b. $|\xi| \ll 1$: $Z(\xi) = i\pi^{\frac{1}{2}} e^{-\xi^2} - 2\xi + 4\frac{\xi^3}{3} - 8\frac{\xi^5}{15} + \dots$

VI. Kinetic Stability

A. Cold Beam Instabilities

1. Cold Beam Distribution: $f_{0s}(\underset{\sim}{v}) = \sum_j n_{js}\, \delta(v_x)\, \delta(v_y)\, \delta(v_z - U_{js})$

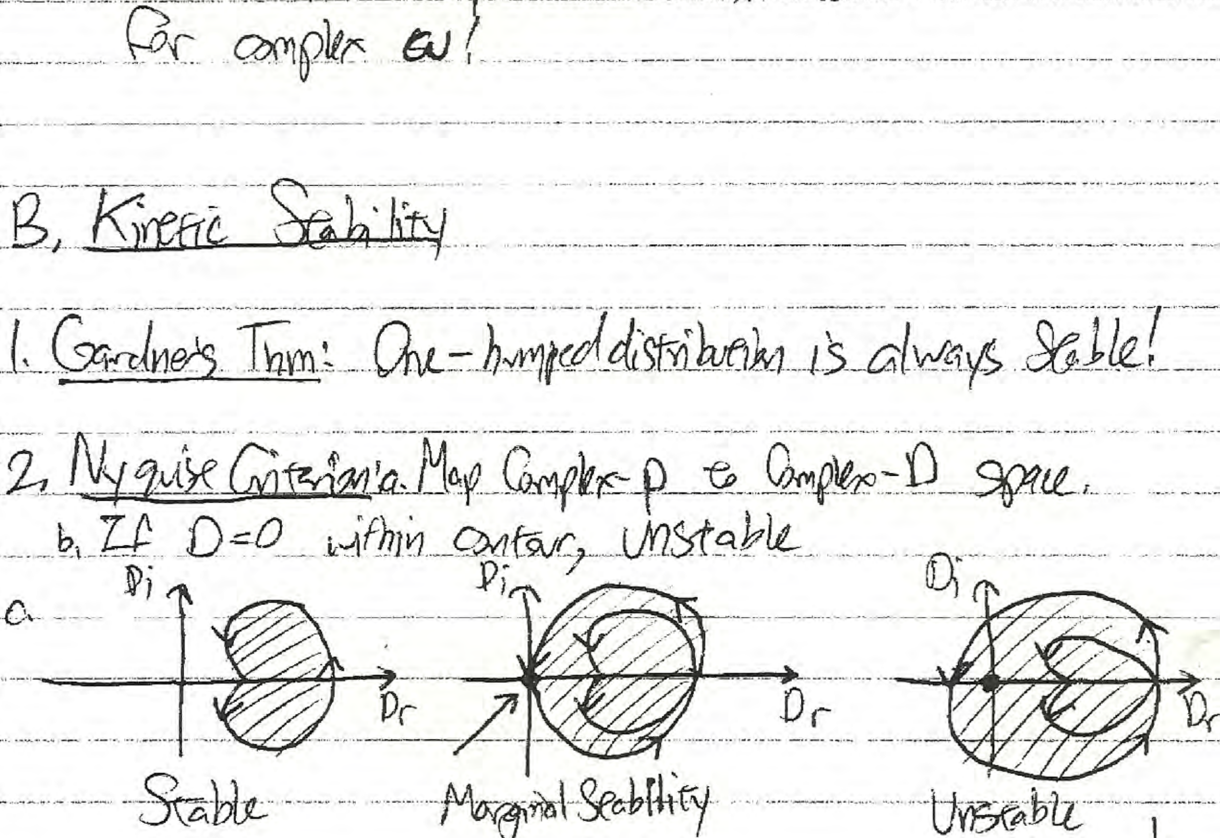
2. Substitute into Dispersion Relation and Solve for complex ω!

B. Kinetic Stability

1. Gardner's Thm: One-humped distribution is always stable!

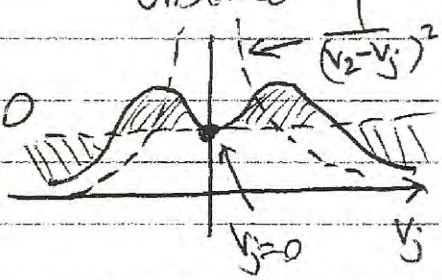
2. Nyquist Criterion: a. Map Complex-p to Complex-D space.
   b. If $D=0$ within contour, Unstable

c.



3. Penrose Condition: $\displaystyle\int_{-\infty}^{\infty} dv_z \, \frac{F_0(v_z) - F_0(v_j)}{(v_z - v_j)^2} > 0$

where $\frac{\partial f_0}{\partial v_z} = 0$!



VII. Quasilinear Theory: Not on Exam

VIII. Plasma Sheaths & Langmuir Probes: Not on Exam