PHYS: 7729 Plasma Physics II

Hawes (1)

Midterm 1 Review:

I. Magnetohydrodynamics (MHD)

A. MHD Approximation

1. Strong Collisions: $\lambda_m \ll L$, $\tau \gg \frac{1}{\nu_{ei}}$
2. Non-relativistic: $V_0^2/c^2 \ll 1$
3. Magnetized: $r_{Li} \ll L$

$\Big\}$ Quasi-neutrality $\sum_s n_s q_s \simeq 0$

B. Derivation from Two-fluid Equations

1. Drop $\underset{\approx}{\Pi}$, viscous terms from $\underset{\approx}{P}$ tensor (Collisional)
2. Isotropic pressure (Collisional)
3. Drop Displacement Current (Non-relativistic)
4. Drop Hall term in Ohm's Law (Magnetized $r_{Li} \gg L$)
5. Adiabatic Equation of State

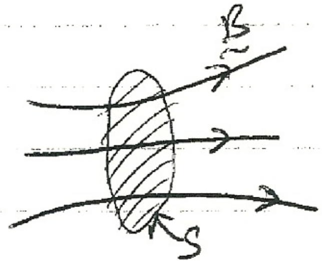
C. MHD Equations

1. Continuity $\quad \frac{\partial \rho}{\partial t} + \underset{\sim}{U} \cdot \nabla \rho = -\rho \nabla \cdot \underset{\sim}{U}$
2. Momentum $\quad \rho \frac{\partial U}{\partial t} + \rho \underset{\sim}{U} \cdot \nabla \rho = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{(\underset{\sim}{B} \cdot \nabla)\underset{\sim}{B}}{\mu_0}$
3. Induction $\quad \frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B}) + \frac{\eta}{\mu_0} \nabla^2 \underset{\sim}{B}$
4. Adiabatic Equation of State $\quad \frac{\partial p}{\partial t} + \underset{\sim}{U} \cdot \nabla p = -\gamma p \nabla \cdot \underset{\sim}{U}$

D. Frozen-In Flux

1. Magnetic Reynolds Number: $Re_M \equiv \frac{\mu_0 L V_0}{\eta}$
2. Limits: a) $Re_M \gg 1 \Rightarrow$ Ideal MHD
   b) $Re_M \ll 1 \Rightarrow$ Diffusive

3. Frozen-In Flux Theorem: (Ideal Limit)



   a. Magnetic Flux $\Phi_B = \int_S \underset{\sim}{B} \cdot d\underset{\sim}{A}$

   b. Magnetic Field lines are frozen to the flow

4. Clebsch Coordinates:
   a. $\underset{\sim}{B} = \nabla \alpha \times \nabla \beta$
   b. $\left.\begin{array}{l} \alpha(\underset{\sim}{x}) = \text{const} \\ \beta(\underset{\sim}{x}) = \text{const} \end{array}\right\}$ Defines field lines $\underset{\sim}{B}(\underset{\sim}{x})$

5. Applications of Frozen-In Flux: $\Phi_B = $ constant as dA changes!

E. MHD Linear Dispersion Relation

1. Know the steps to compute the linear dispersion relation!
   a. Linearize the equations: $\rho = \underbrace{\rho_0}_{\text{uniform, steady}} + \epsilon \rho_1(\underset{\sim}{x})$, $\underset{\sim}{U} = \cancel{\underset{\sim}{U_0}}^{0} + \epsilon \underset{\sim}{U_1}(\underset{\sim}{x})$, etc.
   b. Fourier transform (plane-wave solution)
   $e^{i(\underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t)} \Rightarrow$ $\nabla \Rightarrow i\underset{\sim}{k}$
   $\frac{\partial}{\partial t} \Rightarrow -i\omega$

I. (Continued)

E. (Continued)

c. Eliminate $\rho_1, \underset{\sim}{B}_1, p_1$ in terms of $\underset{\sim}{U}_1$

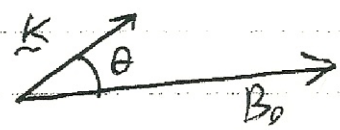
d. Collect to obtain matrix equation

$$\underset{\approx}{M}\cdot\underset{\sim}{U}=0$$

e. Set Determinant $|\underset{\approx}{M}|=0 \Rightarrow$ Yields $D(\omega,\underset{\sim}{k})=0$

f. Solutions for ω (in terms of $\underset{\sim}{k}$ and other parameters) yield the linear wave modes!

F. MHD Wave Modes



1. $\underset{\sim}{k}=k_\perp\hat{\underset{\sim}{x}}+k_\parallel\hat{\underset{\sim}{z}} = k\sin\theta\,\hat{\underset{\sim}{x}}+k\cos\theta\,\hat{\underset{\sim}{z}}$, $\underset{\sim}{B}=B_0\hat{\underset{\sim}{z}}$

2. General Dispersion Relation

$$\underbrace{(\omega^2-k_\parallel^2 v_A^2)}\underbrace{\left[\omega^4-\omega^2k^2(c_s^2+v_A^2)+k_\parallel^2k^2c_s^2v_A^2\right]}=0$$

Alfven Modes, incompressible — Fast and Slow Magnetosonic Waves, compressible

$\omega=\pm k_\parallel v_A$

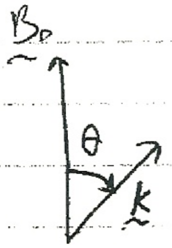
$$\frac{\omega^2}{k^2}=\frac{1}{2}(c_s^2+v_A^2)\pm\frac{1}{2}\sqrt{(c_s^2+v_A^2)-4c_s^2v_A^2\cos^2\theta}$$

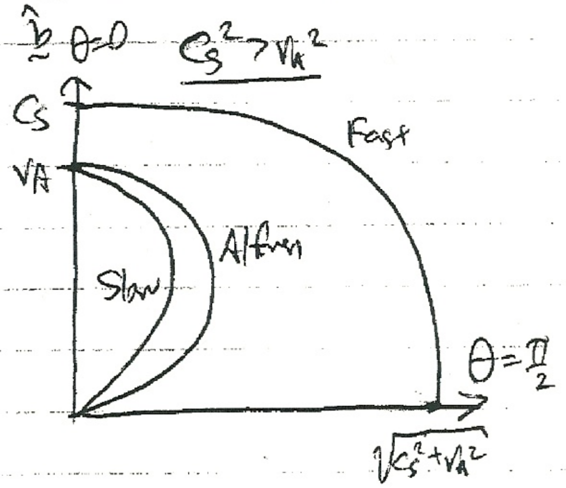
3. Sound speed $c_s^2=\frac{\gamma p_0}{\rho_0}$ — Alfven Speed $v_A^2=\frac{B_0^2}{\mu_0\rho_0}$

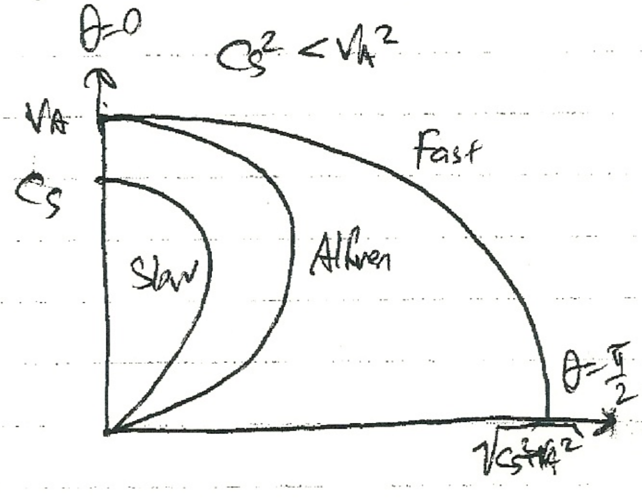
4. a. Fast: Thermal & Magnetic Pressure add
   Slow: Thermal & Magnetic Pressure oppose

I. F. (Continued)

5. <u>Polar Plots</u>: $v_p = \frac{\omega}{k}$ = phase speed







G. <u>Conserved Energy</u>

1. $E = \int d^3\underset{\sim}{x} \left(\frac{1}{2}\rho U^2 + \frac{P}{\gamma - 1} + \frac{B^2}{2\mu_0} \right)$

H. <u>Eigenfunctions of MHD Waves</u>

1. $$\begin{pmatrix} (-\ ) & 0 & (-\ ) \\ 0 & (-\ ) & 0 \\ (-\ ) & 0 & (-\ ) \end{pmatrix} \begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix} = 0$$

2. Take $U_y = 0$, $U_x = U_0$

3. Solve for all other quantities in terms of U_x using the linearized, Fourier-transforms equations

$\Rightarrow \rho_1, \underset{\sim}{U_1}, \underset{\sim}{B_1}, P_1$ in terms of $U_x = U_0$!

4. Plug in ω for particular wave mode.

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II. MHD Equilibrium

1. Force-Balanced Equilibrium: $\nabla p = \underset{\sim}{j} \times \underset{\sim}{B}$

2. Hopf's Thm: Torus is simplest surface such that $\underset{\sim}{B} \cdot \nabla p = 0$.

3. Force-Free Equilibrium: $\underset{\sim}{j} \times \underset{\sim}{B} = 0$

   a. $\Rightarrow \nabla \times \underset{\sim}{B} = \alpha \underset{\sim}{B}$ $\Rightarrow \underset{\sim}{B} \cdot \nabla \alpha = 0$

   surfaces of constant α

4. Cylindrical Equilibria: $\nabla\left(\underset{P_T}{p} + \underset{P_B}{\frac{B^2}{2\mu_0}}\right) = \underset{\text{Mag Tension}}{\frac{(\underset{\sim}{B}\cdot\nabla)\underset{\sim}{B}}{\mu_0}}$

   (Ex. Flux rope, RFP)

   a. $\underset{\sim}{B} = B_\phi(r)\hat{\phi} + B_z(r)\hat{z}$

   b. $\frac{d}{dr}\left(p + \frac{B_\phi^2}{2\mu_0} + \frac{B_z^2}{2\mu_0}\right) = -\frac{B_\phi^2}{\mu_0 r}$

   c. $B_z = 0 \Rightarrow$ Z-pinch

   $B_\phi = 0 \Rightarrow$ Theta-pinch

5. Toroidal Equilibria: (Not on Exam)

   a. Magnetic Flux Coordinates $[\psi(r,z),\ F(\psi)]$

   b. Grad-Shafranov Equation

   $\Rightarrow$ Magnetostatic Equilibria

III. Multiple Timescale Methods

A. General Approach

1. Two timescales: t, $\tau = \epsilon^2 t$

2. $\frac{d}{dt} = \frac{\partial}{\partial t} + \epsilon^2 \frac{\partial}{\partial \tau}$

3. Expand variables: $x = \cancel{x_0}^{0} + \epsilon x_1 + \epsilon^2 x_2 + \ldots$

4. Substitute into equation and multiply out to get all terms

5. Collect terms order by order to obtain
   $O(1)$
   $O(\epsilon)$ equations
   $O(\epsilon^2)$

6. Solve Equations order by order for x_1, x_2, ...
   $\Rightarrow$ set $\epsilon = 1$!

7. NOTE! Every problem is different. I will try to walk you through any problems a give you (Prob #1, HW #3)

III. (Continued)

B. Polarization Drift

1. Slowly varying electric field, $\mathbf{E}(\tau)$ with constant $\mathbf{B} = B_0 \hat{z}$.

2. $$\mathbf{v} = \underbrace{v_\perp \left(\cos(\omega_c t + \phi)\hat{x} - \sin(\omega_c t + \phi)\hat{y}\right)}_{O(1)\text{: Larmor Orbit}} + \underbrace{\frac{\mathbf{E}(\tau)\times\mathbf{B}}{B_0^2}}_{\substack{O(\epsilon) \\ \mathbf{E}\times\mathbf{B}\text{ drift}}} + \underbrace{\frac{1}{\omega_c B_0}\frac{d\mathbf{E}}{dt}}_{\substack{O(\epsilon^2) \\ \text{Polarization Drift, } \mathbf{v}_p}}$$

3. $$\mathbf{v}_p = \frac{1}{\omega_c B}\frac{d\mathbf{E}}{dt} = \frac{m}{qB^2}\frac{d\mathbf{E}}{dt}$$

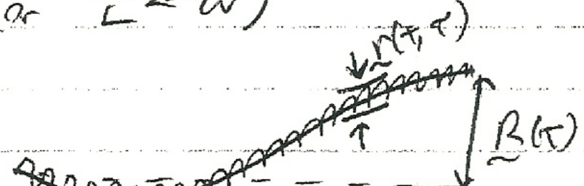
C. Ponderomotive Force

1. Particle motion in rapidly varying electromagnetic field

$$\mathbf{E}(\mathbf{x}, \underset{\text{fast}}{t}, \underset{\text{slow}}{\tau}) = \mathbf{E}_0(\mathbf{x}, \tau)\cos(\omega t - \mathbf{k}\cdot\mathbf{x})$$

2. Ordering: $\epsilon = \frac{V}{L\omega} \ll 1$ (or $\frac{V}{L} \ll \omega$)

3. $$\mathbf{X} = \underset{\text{oscillation center}}{\mathbf{R}(\tau)} + \epsilon\, \mathbf{r}(t, \tau)$$



4. $$\mathbf{F}_{pond} = m\frac{d\mathbf{u}}{dt} = \frac{-q^2}{4m\omega^2}\nabla|\mathbf{E}_0|^2$$

$\mathbf{E}_0(\mathbf{x})$

NOTE: Drop $\cos(\omega t - \mathbf{k}\cdot\mathbf{x})$ in this part!

5. $$\mathbf{F}_{pond} = -\nabla\Phi_{pond} \qquad \Phi_{pond} = \frac{q^2}{4m\omega^2}|\mathbf{E}_0|^2$$

6. Energy Picture: $$\mathcal{E}_{osc} = \frac{1}{2}m u^2 + \Phi_{pond} = \text{constant!}$$

IV. MHD Stability

A. Ideal MHD Equations: Cont $\frac{\partial \rho}{\partial t} + \underset{\sim}{U} \cdot \nabla \rho = -\rho \nabla \cdot U$

Mom. $\rho \left[\frac{\partial \underset{\sim}{U}}{\partial t} + \underset{\sim}{U} \cdot \nabla \underset{\sim}{U} \right] = -\nabla p + \frac{(\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B}}{\mu_0}$

Ind $\frac{\partial B}{\partial t} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B})$

Press. $\frac{\partial p}{\partial t} + \underset{\sim}{U} \cdot \nabla p = -\gamma p \nabla \cdot \underset{\sim}{U}$

2. $E = \int d^3\underset{\sim}{x} \left[\frac{1}{2} \rho U^2 + \frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0} \right]$

B. Linear Force Operator

1. Plasma Equilibrium gives $p_0(\underset{\sim}{x})$, $\underset{\sim}{B}_0(\underset{\sim}{x})$, $\rho_0(\underset{\sim}{x})$ (where $\underset{\sim}{U}_0 = 0$)

2. For a plasma displacement $\underset{\sim}{\xi}_1$

a. $\rho_0 \frac{\partial^2 \underset{\sim}{\xi}_1}{\partial t^2} = \underset{\sim}{F}(\underset{\sim}{\xi}_1)$

b. $\underset{\sim}{F}(\underset{\sim}{\xi}_1) = \nabla \left[\underset{\sim}{\xi}_1 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \underset{\sim}{\xi}_1 \right] + \frac{(\nabla \times \underset{\sim}{B}_0) \times [\nabla \times (\underset{\sim}{\xi}_1 \times \underset{\sim}{B}_0)]}{\mu_0}$
$+ \frac{(\nabla \times [\nabla \times (\underset{\sim}{\xi}_1 \times \underset{\sim}{B}_0)]) \times \underset{\sim}{B}_0}{\mu_0}$

C. Expand Energy in orders of $|\underset{\sim}{\xi}_1|$

1. $\mathcal{O}(|\underset{\sim}{\xi}_1|^0) = \mathcal{O}(1)$ $\qquad E_0 = \int d^3\underset{\sim}{x} \left[\frac{p_0}{\gamma - 1} + \frac{B_0^2}{2\mu_0} \right]$

2. $\mathcal{O}(|\underset{\sim}{\xi}_1|^1)$ $\qquad E_1 = \int d^3\underset{\sim}{x}\, \underset{\sim}{\xi}_0 \left[\nabla p_0 - \frac{(\nabla \times \underset{\sim}{B}_0) \times \underset{\sim}{B}_0}{\mu_0} \right] = 0$

3. $\mathcal{O}(|\underset{\sim}{\xi}_1|^2)$ $\qquad E_2 = \int d^3\underset{\sim}{x} \left[\frac{1}{2} \rho_0 \left| \frac{\partial \underset{\sim}{\xi}_1}{\partial t} \right|^2 \right] + \delta W(\underset{\sim}{\xi}_1, \underset{\sim}{\xi}_1)$
$= \delta K + \delta W$

IV. (Continued)

C. (Continued)

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4. Since $\underset{\sim}{F}(\underset{\sim}{\xi}_1)$ is self-adjoint

$$\int d^3\underset{\sim}{x}\ \underset{\sim}{\eta} \cdot \underset{\sim}{F}(\underset{\sim}{\xi}) = \int d^3\underset{\sim}{x}\ \underset{\sim}{\xi} \cdot \underset{\sim}{F}(\underset{\sim}{\eta})$$

$$\Rightarrow \delta W = \frac{1}{2} \int d^3\underset{\sim}{x}\ \underset{\sim}{\xi} \cdot \underset{\sim}{F}(\underset{\sim}{\xi})$$

D. Normal Modes:

1. $\rho_0 \frac{\partial^2 \underset{\sim}{\xi}_1}{\partial t^2} = \underset{\sim}{F}(\underset{\sim}{\xi}_1) \Rightarrow -\rho_0 \omega_n^2 \underset{\sim}{\xi}_n = \underset{\sim}{F}(\underset{\sim}{\xi}_n)$

   a. $\omega_n^{2\,*} = \omega_n^2 \Rightarrow \omega_n^2$ is real

   b. $\int d^3\underset{\sim}{x}\ \underset{\sim}{\xi}_n \cdot \underset{\sim}{\xi}_m = \delta_{nm} \Rightarrow$ orthonormal displacements!

E. Energy Principle

1. $E_2 = \delta K + \delta W$

2. $\delta W \geq 0 \Rightarrow$ Necessary & sufficient for stability

   $\delta W < 0$ for any $\underset{\sim}{\xi} \Rightarrow$ UNSTABLE

3. Write $\delta W(\underset{\sim}{\xi}, \underset{\sim}{\xi})$ and minimize w.r.t. $\underset{\sim}{\xi}$ (each component)

   If $\delta W(\underset{\sim}{\xi}_{min}, \underset{\sim}{\xi}_{min}) < 0 \Rightarrow$ UNSTABLE

4. $$\delta W = \frac{1}{2} \int d^3\underset{\sim}{x} \Bigg[\underbrace{\frac{|\nabla \times (\underset{\sim}{\xi} \times \underset{\sim}{B}_0)|^2}{\mu_0}}_{\text{Mag tension \& compression}} + \underbrace{\gamma p_0 |\nabla \cdot \underset{\sim}{\xi}|^2}_{\text{Thermal Compression}}$$
$$- \underbrace{\underset{\sim}{\xi}^* \cdot \underset{\sim}{j}_0 \times [\nabla \times (\underset{\sim}{\xi} \times \underset{\sim}{B}_0)]}_{\text{"Kink drive"}} - \underbrace{\underset{\sim}{\xi}^* \cdot \nabla(\underset{\sim}{\xi} \cdot \nabla p_0)}_{\text{"Interchange drive"}} \Bigg]$$

potentially destabilizing