PHYS:7729

Review #2

Howes ①

I. General Comments.

A. Where use

1. 2 page formula summary sheet
2. NRL Plasma Formulary

B. Topics to be covered

1. Lectures #14–21,
2. Homework #4–7

II. Review

A. Kinetic Theory

1. Boltzmann Equation: $\frac{\partial f_s}{\partial t} + \underset{\sim}{v} \cdot \nabla f_s + \frac{q_s}{m_s}(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} = \left(\frac{\partial f_s}{\partial t}\right)_{coll}$ $= 0 \Rightarrow$ Vlasov Equation

   where $f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)$ is 6-D distribution fnc.

2. $\nabla \cdot \underset{\sim}{E} = \frac{\rho_q}{\epsilon_0}$ $\qquad \rho_q = \sum_s \int d^3\underset{\sim}{v}\, q_s f_s$

   $\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}$ $\qquad \underset{\sim}{j} = \sum_s \int d^3\underset{\sim}{v}\, q_s \underset{\sim}{v} f_s$

   $\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{j} + \mu_0 \epsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t}$

   $\nabla \cdot \underset{\sim}{B} = 0$

3. Maxwellian Distribution $f_{s_m}(\underset{\sim}{x}, \underset{\sim}{v}, t) = \frac{n_s(\underset{\sim}{x}, t)}{\pi^{3/2} v_{ts}^3(\underset{\sim}{x}, t)} e^{-\frac{m_s |\underset{\sim}{v} - \underset{\sim}{U}_s(\underset{\sim}{x}, t)|^2}{2 T_s(\underset{\sim}{x}, t)}}$

   $\Rightarrow f_{s_m}(\underset{\sim}{v}) = \frac{n_{s0}}{\pi^{3/2} v_{ts}^3} e^{-\frac{v^2}{v_{ts}^2}}$ where $v_{ts} = \sqrt{\frac{2T_s}{m_s}}$

4. Moments: $n_{s0} = \int d^3\underset{\sim}{v}\, f_{s_m}(\underset{\sim}{v})$

   $\frac{3}{2} n_{s0} T_{s0} = \int d^3\underset{\sim}{v}\, \frac{1}{2} m_s v^2 f_{s_m}(\underset{\sim}{v})$

5. Reduced Distribution Function: $F_s(v_z) = \frac{1}{n_{s0}} \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y\, f_{s_m}(\underset{\sim}{v})$

   a. $\int_{-\infty}^{\infty} dv_z\, F_s(v_z) = 1$

Review #2 (Continued) Howes ③

II. (Continued)

B. Electrostatic Waves in Unmagnetized Plasma

1. $\mathbf{E} = -\nabla\phi$

2. Linearized a. $\frac{\partial f_{s1}}{\partial t} + \mathbf{v}\cdot\nabla f_{s1} - \frac{q_s}{m_s}\nabla\phi\cdot\frac{\partial f_{s0}}{\partial \mathbf{v}} = 0$

b. $-\nabla^2\phi_1 = \frac{1}{\epsilon_0}\sum_s \int d^3\mathbf{v}\, q_s f_{s1}$

3. Fourier Transform Solution: $\frac{\partial}{\partial t} \Rightarrow -i\omega \quad \nabla \Rightarrow i\mathbf{k}$

a. $D(\mathbf{k},\omega) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2}\int_{-\infty}^{\infty} dv_z \frac{\partial F_{s0}/\partial v_z}{(v_z - \frac{\omega}{k})} = 0$

b. Only valid where $F_{s0}=0$ and $\frac{\partial F_{s0}}{\partial v_z}=0$ at $v_z = \frac{\omega}{k}$.

c. Can integrate by parts to get $D(\mathbf{k},\omega) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2}\int_{-\infty}^{\infty} dv_z \frac{F_{s0}}{(v_z-\frac{\omega}{k})^2} = 0$

d. For $v_z \ll \frac{\omega}{k}$, we can expand denominator $\frac{1}{(v_z-\frac{\omega}{k})^2} \simeq \frac{k^2}{\omega^2}\left(1 + 2\left(\frac{kv_z}{\omega}\right) + \dots\right)$

4. Laplace Transforms

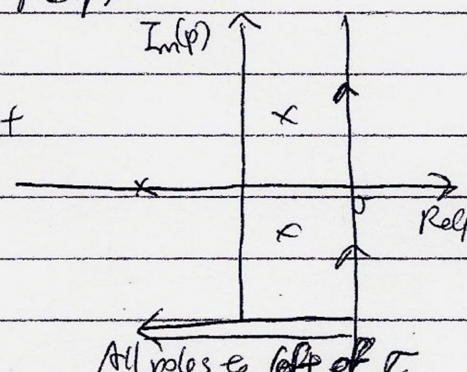
a. Landau solved the problem by using Laplace-Fourier Transforms, thus handling the initial value problem and determining growth/damping

b. DEF: $\tilde{f}(p) = \int_0^\infty dt\, f(t) e^{-pt}$ $\qquad Re(p) > 0$

$p = \gamma - i\omega$

c. INVERSE: $f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dp\, \tilde{f}(p) e^{pt}$

d. $\tilde{f'}(p) = p\tilde{f}(p) - f(0)$



All poles to left of σ.

5. Contour Integration. a. Analytic Function $f(z)$ → Derivative everywhere.

b. Deformation of Paths

c. Cauchy Integral Formula: $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-z_0}$

d. Residue Theorem: $\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{\text{Res}}[f(z)]$

Review #2 (Continued) Howes ③

II B. (Continued)

6. Laplace-Fourier Solution of Simple System

a. This is important!

7. Laplace-Fourier Solution for Electrostatic Plasma Waves

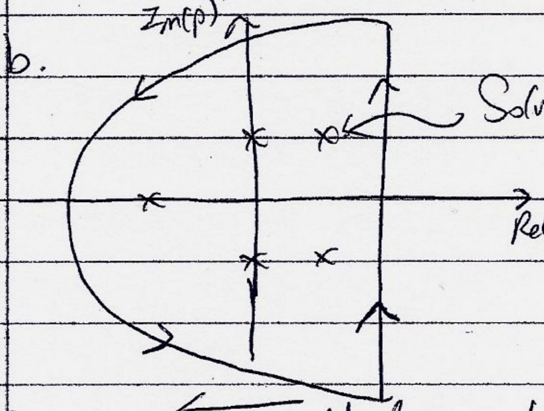
a. $$\phi_1(\underline{k},p) = \frac{N(\underline{k},p)}{D(\underline{k},p)}$$

where $$N(\underline{k},p) = -i \sum_s \frac{q_s n_0}{\epsilon_0 k^3} \int_{-\infty}^{\infty} dv_z \frac{F_s(0)}{v_z - ip/k}$$

and Dispersion Relation is given by

$$D(\underline{k},p) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\partial F_{s0}/\partial v_z}{v_z - \frac{ip}{k}} = 0 \Rightarrow \text{Solutions!}$$

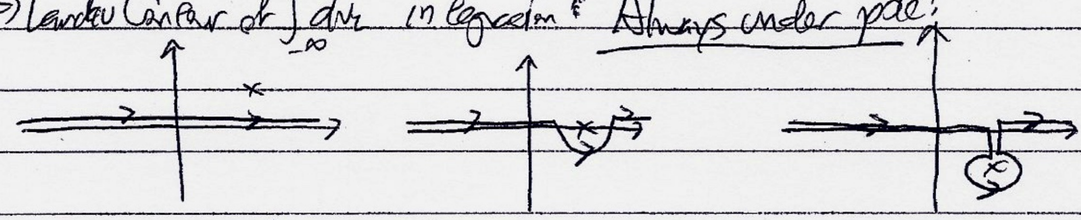
b.



Solution are related to poles of $\phi(\underline{k},p)$

c. Need to analytically continue $\phi(\underline{k},p)$ to $Re(p)<0$ plane.

$\Rightarrow$ Landau Contour of $\int_{-\infty}^{\infty} dv_z$ in equation: Always under pole!



d. Evaluate using Residue Theorem

8. Cauchy Distribution: $F_{0e}(v_z) = \frac{c}{\pi} \frac{1}{c^2 + v_z^2}$

9. Weak Growth Rate Approximation: $D_r(k, \omega_r) = 0 \Rightarrow \omega_r$

$$\gamma = \frac{-D_i(k,\omega_r)}{\partial D_r(k,\omega_r)/\partial \omega_r}$$

Review #2 (Continued)

II. B. (Continued)

10. Plemelj Relation

a. $$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} dx \frac{f(x)}{x-(x_0 \pm i\epsilon)} = P\int_{-\infty}^{\infty} dx \frac{f(x)}{x-x_0} \pm i\pi f(x_0)$$

b. Principal Value $$P\int_{-\infty}^{\infty} \frac{g(x)}{x-x_0} dx = \lim_{\delta \to 0} \left[\int_{-\infty}^{x_0-\delta} dx \frac{g(x)}{x-x_0} + \int_{x_0+\delta}^{\infty} dx \frac{g(x)}{x-x_0} \right]$$

11. For a general equilibrium, we find

$$\gamma = \pi \frac{k}{|k|} \frac{1}{\frac{\partial D_r(k,\omega_r)}{\partial \omega_r}} \sum_s \frac{\omega_{ps}^2}{k^2} \left.\frac{\partial f_{0s}}{\partial v_z}\right|_{v_z = \frac{\omega}{k}}$$

12. Plasma Dispersion Function:

a. $$Z(\xi_s) = \int_C \frac{dz}{\pi^{1/2}} \frac{e^{-z^2}}{z-\xi_s} \qquad \xi_s = \frac{ip}{k v_{ts}} = \frac{\omega}{k v_{ts}} - i\frac{\gamma}{k v_{ts}}$$

b. $|\xi| \gg 1$: $$Z(\xi) \simeq i\pi^{1/2} e^{-\xi^2} - \frac{1}{\xi} - \frac{1}{2\xi^3} - \frac{3}{4\xi^5} - \dots$$

c. $|\xi| \ll 1$: $$Z(\xi) \simeq i\pi^{1/2} e^{-\xi^2} - 2\xi + \frac{4\xi^3}{3} - \frac{8\xi^5}{15} + \dots$$

B. For ion and electron plasma, (Maxwellian) a. Langmuir Waves $\omega^2 = \omega_p^2 + \frac{3}{2}k^2 v_{te}^2$

b. Ion Acoustic Waves $\omega^2 = k^2 c_i^2 \qquad c_i^2 = \frac{T_e}{m_i}$

C. Cold Beam Instabilities

1. Cold Beam Distribution: $f_{0s}(\mathbf{v}) = \sum_j n_{js}\, \delta(v_x)\, \delta(v_y)\, \delta(v_z - V_{js})$

2. Two-Stream Instability: $n_1 = n_2 = n_0 \quad V_1 = -V_2 = V$

a. $$\omega^2 = \omega_p^2 + k^2V^2 \pm \sqrt{\omega_p^4 + 4\omega_p^2 k^2 V^2}$$

b. $\Rightarrow k < k_c = \sqrt{2}\frac{\omega_p}{V}$ for Instability.

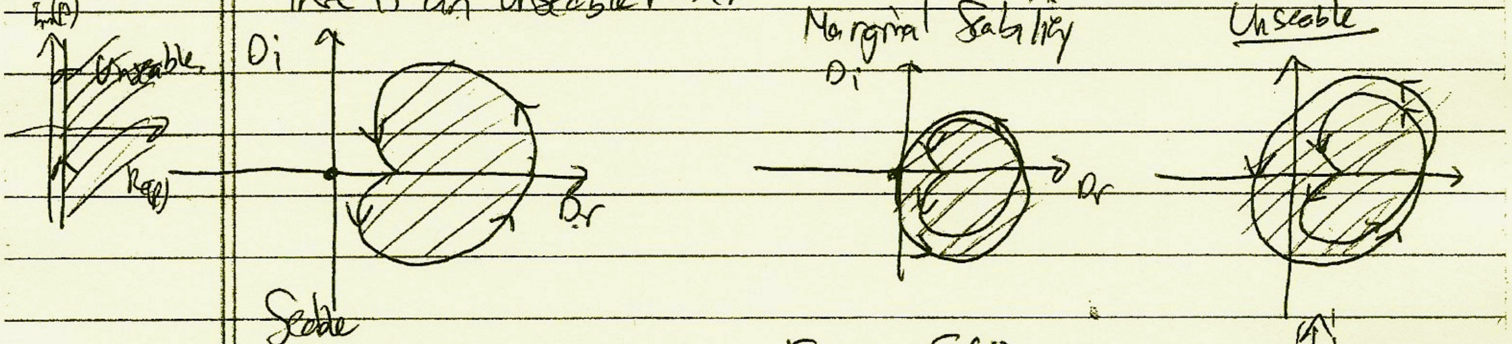
D. Kinetic Stability:

1. Gardner's Theorem: Single-humped velocity distribution is always stable.

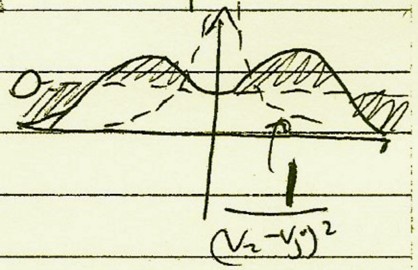
Review #2 (Continued)

II D. (Continued)

2. Nyquist Criterion: Map marginal stability criterion from complex-p space to complex D-space. If $D=0$ falls within contour there is an unstable root.



3. Penrose Condition: $\int_{-\infty}^{\infty} dv_z \frac{F_0(v_z)-F_0(v_j)}{(v_z-v_j)^2} > 0$



III. A. Quasilinear Theory:

1. Mean & Fluctuating Values: a. Mean $\langle f_s(v_z,t)\rangle = \frac{1}{2L}\int_{-L}^{L} dz\, f_s(z,v_z,t)$

b. $f_s(z,v_z,t) = \langle f_s(v_z,t)\rangle + \epsilon f_{s1}(z,v_z,t)$

2. Resulting Equations for Electrostatic Waves:

a. $\frac{\partial \langle f_s\rangle}{\partial t} = \epsilon^2 \frac{q_s}{m_s}\frac{\partial}{\partial v_z}\left\langle f_{s1}\frac{\partial \phi_1}{\partial z}\right\rangle$

Approximation $\to 0$

b. $\epsilon\left(\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z}\right) f_{s1} = \epsilon \frac{q_s}{m_s}\frac{\partial \phi_1}{\partial z}\frac{\partial\langle f_s\rangle}{\partial v_z} + \cancel{\epsilon^2 \frac{q_s}{m_s}\frac{\partial}{\partial v_z}\left[f_{s1}\frac{\partial\phi_1}{\partial z} - \left\langle f_{s1}\frac{\partial \phi_1}{\partial z}\right\rangle\right]}$

c. $\epsilon \frac{\partial^2\phi_1}{\partial z^2} = -\epsilon \sum_s \frac{q_s}{\epsilon_0}\int_{-\infty}^{\infty} dv_z\, f_{s1}$

3. Quasilinear Diffusion Equation: a. NOTE: $\hat{\phi}_1(k) = \hat{\phi}_1^*(-k)$ $\quad \omega_r(k,\tau) = -\omega_r(-k,\tau)$ $\quad \gamma(k,\tau) = \gamma(-k,\tau)$

b. $\frac{\partial\langle f_s\rangle}{\partial t} = \frac{\partial}{\partial v_z}\left[D_q(v_z,t,\tau)\frac{\partial\langle f_s\rangle}{\partial v_z}\right]$

$$D_q(v_z,t,\tau) = \frac{2}{\epsilon_0}\left(\frac{q_s}{m_s}\right)^2\int_{-\infty}^{\infty} dk\, \frac{\mathcal{E}(k,t,\tau)\,\gamma(k,\tau)}{[\omega_r(k,\tau)-kv_z]^2 + [\gamma(k,\tau)]^2}$$

Review #2 (Continued)

Hawes (6)

II. A. (Continued)

3 c. $\frac{\partial \mathcal{E}(k,t,\tau)}{\partial t} = 2\gamma(k,\tau)\, \mathcal{E}(k,t,\tau)$

d. $D(k,\omega) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\frac{\partial \langle f_s \rangle}{\partial v_z}}{v_z - \frac{\omega}{k}} = 0$

e. Must evolve $\langle f_s(v_z,\tau)\rangle$, $\mathcal{E}(k,t,\tau)$, and $\gamma(k,\tau)$ self-consistently.

4. Application: