

Lecture #0 The Two Fluid Equations

Hanes ①

I. Simplifications of the Pressure Tensor $\underline{\underline{P}}_s$

A. Review

1. By taking the first moment of the Plasma Kinetic Equation, we derive the Fluid Momentum Equation

$$n_s m_s \left[\frac{\partial \underline{U}_s}{\partial t} + \underline{U}_s \cdot \nabla \underline{U}_s \right] = -\nabla \cdot \underline{\underline{P}}_s + n_s q_s (\underline{E} + \underline{U}_s \times \underline{B}) + \underline{F}_{D_s}$$

↑
Collisional Drag

a. Convective Derivative: $\frac{d \underline{U}_s}{dt} \equiv \frac{\partial \underline{U}_s}{\partial t} + \underline{U}_s \cdot \nabla \underline{U}_s$

The total derivative along the trajectory of a fluid element.

B. Isotropic Pressure

1. The pressure tensor includes both pressure and viscosity.

$$\underline{\underline{P}}_s = p_s \underline{\underline{I}} + \underline{\underline{\Pi}}_s = \begin{bmatrix} p_s & 0 & 0 \\ 0 & p_s & 0 \\ 0 & 0 & p_s \end{bmatrix} + \underline{\underline{\Pi}}_s$$

$p_s = \frac{1}{3} \text{Tr}[\underline{\underline{P}}_s]$

↑
Viscosity Tensor

2. In this case $-\nabla \cdot \underline{\underline{P}}_s \approx -\nabla p_s - \nabla \cdot \underline{\underline{\Pi}}_s$

3. The viscosity leads to the transport of momentum.

a. $\underline{\underline{\Pi}}_s \sim \mathcal{O}\left(\frac{\lambda_m}{L} p_s\right)$ where L is scale length of fluid variables $n_s(\underline{x}, t)$, $\underline{U}_s(\underline{x}, t)$, $T_s(\underline{x}, t)$.

b. In a collisional plasma, $\lambda_m \ll L$, so the viscous term is small compared to the pressure term.

c. Thus, the viscosity term is then dropped in a collisional, fluid plasma.

$$\Rightarrow -\nabla \cdot \underline{\underline{P}}_s \approx -\nabla p_s$$

4. For a Maxwellian distribution $f_{sM}(\underline{x}, \underline{v}, t)$, $\underline{\underline{P}}_s = \int d^3v m_s (\underline{v} - \underline{U}_s)(\underline{v} - \underline{U}_s) f_{sM}(\underline{x}, \underline{v}, t)$

gives $\underline{\underline{P}}_s = \begin{bmatrix} n_s k T_s & 0 & 0 \\ 0 & n_s k T_s & 0 \\ 0 & 0 & n_s k T_s \end{bmatrix}$. Thus $\boxed{p_s = n_s k T_s}$

I. (Continued)

C. Anisotropic Distribution

1. For a Bi-Maxwellian distribution $f_{BM}(\underline{x}, \underline{v}, t)$, we can integrate over velocity space to obtain

$$\underline{P}_s = \begin{bmatrix} P_{\perp s} & 0 & 0 \\ 0 & P_{\parallel s} & 0 \\ 0 & 0 & P_{\parallel s} \end{bmatrix} \quad \text{where} \quad \begin{aligned} P_{\perp s} &= n_s k T_{\perp s} \\ P_{\parallel s} &= n_s k T_{\parallel s} \end{aligned}$$

II. Solutions to the Closure Problem:

A. Introduction

- As stated at the end of Lecture 3, the procedure of taking the n^{th} moment of the Plasma Kinetic Equation also involves the $(n+1)^{\text{st}}$ moment due to the term $\underline{v} \cdot \nabla f_s$.
- To close the system of equations, we need to specify the $(n+1)^{\text{st}}$ moment in terms of the first n moments.
 - Ex: If we can specify $\underline{P}_s = f(n_s, \underline{U}_s)$, then we can close the system of equations.
- Generally, this can only be done in an ad hoc manner.
 - Usually close the system with a physically motivated Equation of State.

B. Cold Plasma Equation of State

- If we assume the temperature $T_s \rightarrow 0$, then $\underline{P}_s \rightarrow 0$.
- Neglecting collisional drag term, we obtain

$$m_s \frac{d\underline{U}_s}{dt} = q_s (\underline{E} + \underline{U}_s \times \underline{B})$$

- This is the same form as the Lorentz Force Law, but $\underline{U}_s(\underline{x}, t)$ is the fluid velocity and $\frac{d\underline{U}_s}{dt} = \frac{\partial \underline{U}_s}{\partial t} + \underbrace{\underline{U}_s \cdot \nabla \underline{U}_s}_{\text{Convective Derivative}} \underbrace{\quad}_{\text{Introduces a nonlinear term.}}$

II. (Continued)

C. Adiabatic Equation of State

1. From thermodynamics, if no heat flow during a compression (adiabatic motion), the

$$pV^\gamma = \text{constant}$$

where V = volume of gas

$$\gamma = \text{adiabatic index} = \frac{C_p}{C_v}$$

2. For this to be valid:

a. Motion must be rapid to prevent heat flow (adiabatic)

b. Collisions must maintain Local Thermodynamic Equilibrium

3. Ex: Sound waves in a gas are typically adiabatic.

4. In a collisionless plasma, the adiabatic equation of state is often not strictly valid, but it is nevertheless commonly used.

5. For a collisional plasma, $p_s \rightarrow p_s I$ (isotropic pressure).

6. Adiabatic Equation of State

$$\frac{d}{dt} \left(\frac{p_s}{n_s^\gamma} \right) = 0$$

← Expresses the Conservation of Entropy

where $p_s = n_s k T_s$

and $\gamma = \frac{d+2}{d}$ for a gas with d degrees of freedom.

a. For a monatomic "gas" \rightarrow a plasma of H^+ ions and e^- electrons,

$$\gamma = \frac{5}{3}.$$

D. Double Adiabatic Equation of State

Also known as Chew-Goldberger-Law (CGL) Equation of State.

1. In a magnetized plasma, the timescale to transfer momentum from parallel to perpendicular directions (and vice-versa) is short long.

\Rightarrow Bi-Maxwellian distribution $f_{sBM}(T_\perp, T_\parallel)$.

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II (Continued)

2. Double Adiabatic Equations of State (CGL)

$$\frac{d}{dt} \left(\frac{P_{\perp}}{n_s B} \right) = 0$$

$$\frac{d}{dt} \left(\frac{P_{\parallel} B^2}{n_s^3} \right) = 0$$

3. a. Since $P_{\perp} = n_s k T_{\perp}$, $\frac{P_{\perp}}{n_s B} = \frac{n_s k T_{\perp}}{n_s B}$

b. But $v_{T_s}^2 = \frac{2kT_s}{m_s}$, so $kT_s = \frac{1}{2} m_s v_{T_s}^2 \Rightarrow \frac{P_{\perp}}{n_s B} = \frac{m_s v_{T_s}^2}{2B} = \mu$

c. Thus, we see the first of the equations is just a fluid form of conservation of first adiabatic invariant, the magnetic moment μ .
(Here v_{T_s} is replaced by $v_{T_{\perp s}}$).

III. The Two Fluid Equations

1. Many phenomena in plasma physics can be described as two interpenetrating fluids, an ion fluid & an electron fluid.

2. Neglecting viscosity ($\eta \rightarrow 0$) and taking Adiabatic Eq. of State ($Q_s \rightarrow 0$),

Continuity Equation $\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \underline{U}_s) = 0$

Momentum Equation $m_s n_s \left[\frac{\partial \underline{U}_s}{\partial t} + \underline{U}_s \cdot \nabla \underline{U}_s \right] = -\nabla p_s + q_s n_s (\underline{E} + \underline{U}_s \times \underline{B}) + \underline{F}_{ps}$

Adiabatic equation of State $\frac{\partial}{\partial t} \left(\frac{P_s}{n_s^{\gamma}} \right) + \underline{U}_s \cdot \nabla \left(\frac{P_s}{n_s^{\gamma}} \right) = 0$

Charge Density $\rho = \sum_s n_s q_s$

$$\nabla \cdot \underline{E} = \frac{\rho}{\epsilon_0}$$

Maxwell's

Current Density $\underline{j} = \sum_s n_s q_s \underline{U}_s$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

Equations

$$\nabla \times \underline{B} = \mu_0 \underline{j} + \mu_0 \epsilon_0 \frac{\partial \underline{E}}{\partial t}$$

$$\nabla \cdot \underline{B} = 0$$

16 Equations for 16 unknowns: $n_i, n_e, \underline{U}_i, \underline{U}_e, p_i, p_e, \underline{E}, \underline{B}$

IV. Generalized Ohm's Law

A. Small Electron Mass Approximation

1. An approximation frequently used in plasma physics is $m_e \ll m_i$

2. For Hydrogen, $\frac{m_i}{m_e} = 1836$, so this approximation is always well satisfied

B. Electron Momentum Equation

$$1. n_e m_e \left[\frac{\partial \underline{U}_e}{\partial t} + \underline{U}_e \cdot \nabla \underline{U}_e \right] = -\nabla p_e + e n_e \left[\underline{E} + \underline{U}_e \times \underline{B} \right] + \underline{F}_{De}$$

2. a. Because $m_i \gg m_e$, the ion dominate the flow velocity

$$\underline{U} = \frac{m_i \underline{U}_i + m_e \underline{U}_e}{m_i + m_e} \approx \underline{U}_i$$

b. But, the electrons are much more mobile, so $\underline{U}_i - \underline{U}_e$ gives rise to a current

c.

So, we can let the ion velocity $\underline{U}_i \approx \underline{U}$

$$\underline{j} = e n_i \underline{U}_i - e n_e \underline{U}_e = e n_e (\underline{U}_i - \underline{U}_e) \approx + e n_e (\underline{U} - \underline{U}_e)$$

assume quasi-neutral plasma $n_i = n_e$

$$\Rightarrow \underline{U}_e = \underline{U} - \frac{\underline{j}}{e n_e}$$

3. Because the electron mass is small, we can neglect the inertial terms on LHS as small, leaving

$$a. 0 = -\nabla p_e - e n_e \left[\underline{E} + \underline{U}_e \times \underline{B} \right] + \underline{F}_{De}$$

$$b. \text{Substituting for } \underline{U}_e, \quad -\frac{\nabla p_e}{e n_e} + \frac{\underline{F}_{De}}{e n_e} = \underline{E} + \underline{U} \times \underline{B} - \frac{\underline{j} \times \underline{B}}{e n_e}$$

$$c. \quad \underline{E} + \underline{U} \times \underline{B} = -\frac{\nabla p_e}{e n_e} + \frac{\underline{j} \times \underline{B}}{e n_e} + \frac{\underline{F}_{De}}{e n_e}$$

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IV. B. (Continued)

4. But from Lecture #11, we know the collisional drag term

$$a. \underline{F}_{pe} = m_e n_e \underline{v}_e (\underline{v}_i - \underline{v}_e) = \frac{m_e \underline{v}_e}{e} \left[e n_e (\underline{v}_i - \underline{v}_e) \right] = \frac{m_e \underline{v}_e}{e} \underline{j}$$

b. Thus $\frac{\underline{F}_{pe}}{e n_e} = \frac{m_e \underline{v}_e}{e^2 n_e} \underline{j} = \eta \underline{j}$ where $\eta = \frac{m_e \underline{v}_e}{e^2 n_e}$

5. Generalized Ohm's Law

$$\underline{E} + \underline{v} \times \underline{B} = \underbrace{\eta \underline{j}}_{\text{Resistivity}} + \underbrace{\frac{\underline{j} \times \underline{B}}{e n_e} - \frac{\nabla p_e}{e n_e}}_{\text{Hall Terms}}$$