Lecture #1 The Magnetohydrodynamic (MHD) Equations

Howes ①

I. Single Fluid Description of a Magnetized Plasma

A. Introduction

1. Last time we derived the Two Fluid Equations, describing the plasma as interpenetrating ion and electron fluids.
2. Often, we are simply interested in the macroscopic behavior of a plasma, that is, at large ~~length~~ scales and on long timescales.
3. Magnetohydrodynamics, or MHD, is an even more simple, single fluid description of the plasma.
4. MHD is certainly the most widely used plasma description, chosen frequently due to its simplicity. It is often used even in parameter regimes where it is not strictly valid.
5. We'll discuss the limits of validity for MHD in detail here.

II. Derivation of the MHD Equations

A. The MHD Approximation

1. System Size and Observation Time

$$\left.\begin{array}{l} L \equiv \text{System size} \\ \tau \equiv \text{observation time} \end{array}\right\} \text{Characteristic Velocity } V_0 \equiv \frac{L}{\tau}$$

2. We will see that the MHD Equations require the following conditions:
   a. Strong Collisions $\lambda_{m} \ll L$ or $\tau \gg \frac{1}{\nu_{ei}}$
   b. Non-relativistic $V_0^2/c^2 \ll 1$
   c. Quasineutrality $\sum_s n_s q_s \simeq 0$
   d. Magnetized $r_{Li} \ll L$

3. As we derive the MHD Equations, we'll explore these required conditions in more detail.
4. In the following derivation, we'll consider a fully ionized Hydrogen plasma, so $m_i = m_p$, $q_i = e = -q_e$.

Lecture #1 (Continued) Howes ②

II. (Continued)

B. The Continuity Equation

1. Multiply Two Fluid Continuity Equations by m_s and sum over species:

a. $$\sum_s m_s \frac{\partial(n_s)}{\partial t} + \sum_s m_s \nabla\cdot(n_s \underset{\sim}{U_s}) = 0$$

b. $$\frac{\partial}{\partial t}\left(\sum_s n_s m_s\right) + \nabla\cdot\left(\sum_s n_s m_s \underset{\sim}{U_s}\right) = 0$$

2. Define: Mass Density $\boxed{\rho \equiv \sum_s n_s m_s}$ ← NOTE: This is not the charge density; we will use ρ_q to denote charge density.

Fluid Velocity $\boxed{\underset{\sim}{U} \equiv \frac{1}{\rho}\sum_s n_s m_s \underset{\sim}{U_s}}$ ← mass weighted

3. Thus $$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underset{\sim}{U}) = 0}$$ Continuity Equation

C. The Momentum Equation

1. ~~Multiply by m_s and~~ Sum over species.

$$\sum_s \frac{\partial}{\partial t}(n_s m_s \underset{\sim}{U_s}) + \sum_s \nabla\cdot(n_s m_s \underset{\sim}{U_s}\underset{\sim}{U_s}) = -\sum_s \nabla\cdot \underset{\approx}{P_s} + \sum_s n_s q_s(\underset{\sim}{E} + \underset{\sim}{U_s}\times \underset{\sim}{B}) + \sum_s \underset{\sim}{F}_{Ds}$$

2. Collisional Drag Term: When summed over all species, the collisional drag term must conserve momentum, so $\sum_s \underset{\sim}{F}_{Ds} = 0$.

3. Pressure Term: Define a new pressure tensor in terms of fluid velocity $\underset{\sim}{U}$

a. $$\underset{\approx}{P}_{0s} = m_s \int d^3\underset{\sim}{v}\,(\underset{\sim}{v}-\underset{\sim}{U})(\underset{\sim}{v}-\underset{\sim}{U}) f_s(\underset{\sim}{x},\underset{\sim}{v},t)$$

b. This is related to the usual pressure tensor by $\underset{\approx}{P}_{0s} = \underset{\approx}{P}_s + n_s m_s \underset{\sim}{W_s}\underset{\sim}{W_s}$

where $\underset{\sim}{W_s} = \underset{\sim}{U_s} - \underset{\sim}{U}$

c. Thus, we get $$-\nabla\cdot\left(\sum_s \underset{\approx}{P_s}\right) = -\nabla\cdot\left(\sum_s \underset{\approx}{P}_{0s}\right) + \nabla\cdot\sum_s n_s m_s \underset{\sim}{W_s}\underset{\sim}{W_s}$$

II C. (Continued)

4. Lorentz Force Term: $\sum_s n_s q_s \underset{\sim}{E} + \sum_s n_s q_s \underset{\sim}{U_s} \times \underset{\sim}{B}$

a. Define: Charge Density $\boxed{\rho_q \equiv \sum_s n_s q_s}$

Current Density $\boxed{\underset{\sim}{j} = \sum_s n_s q_s \underset{\sim}{U_s}}$

b. Thus $= \rho_q \underset{\sim}{E} + \underset{\sim}{j} \times \underset{\sim}{B}$

5. LHS: Substituting $\underset{\sim}{U_s} = \underset{\sim}{U} + \underset{\sim}{W_s}$ gives

a.
$$\frac{\partial}{\partial t}\underbrace{\left(\sum_s n_s m_s \underset{\sim}{U_s}\right)}_{\rho \underset{\sim}{U}} + \nabla\cdot\left(\rho \underset{\sim}{U}\underset{\sim}{U}\right) + \nabla\cdot\left(\sum_s n_s m_s \underset{\sim}{U}\underset{\sim}{W_s}\right) + \nabla\cdot\left(\sum_s n_s m_s \underset{\sim}{W_s}\underset{\sim}{U}\right) + \nabla\cdot\left(\sum_s \rho_s \underset{\sim}{W_s}\underset{\sim}{W_s}\right)$$

b. Note: $\sum_s n_s m_s \underset{\sim}{U}\underset{\sim}{W_s} = \underset{\sim}{U}\left(\sum_s n_s m_s \underset{\sim}{W_s}\right) = \underset{\sim}{U}\left(\sum_s n_s m_s (\underset{\sim}{U_s} - \underset{\sim}{U})\right) = \underset{\sim}{U}(\cancel{\underset{\sim}{U}} - \underset{\sim}{U}) = 0$

Thus 3rd & 4th terms $= 0$.

c. Thus
$$\frac{\partial}{\partial t}(\rho \underset{\sim}{U}) + \nabla\cdot(\rho \underset{\sim}{U}\underset{\sim}{U}) + \nabla\cdot\left(\sum_s \rho_s \underset{\sim}{W_s}\underset{\sim}{W_s}\right)$$

6. Putting everything together, the $\nabla\cdot\left(\sum_s \rho_s \underset{\sim}{W_s}\underset{\sim}{W_s}\right)$ terms cancel to give

$$\boxed{\frac{\partial(\rho \underset{\sim}{U})}{\partial t} + \nabla\cdot(\rho \underset{\sim}{U}\underset{\sim}{U}) = -\nabla\cdot \underset{\approx}{P_0} + \rho_q \underset{\sim}{E} + \underset{\sim}{j} \times \underset{\sim}{B}}$$

where $\underset{\approx}{P_0} = \sum_s \underset{\approx}{P_{0s}}$

7. Isotropic Pressure:

a. ~~The~~ Pressure will become isotropic on collision time with ions.
Recall from Lect #11, $\nu_{ii} = \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \nu_{ei}$

b. So, for times $\tau \gg \frac{1}{\nu_{ii}} \sim \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \frac{1}{\nu_{ei}}$, Isotropic pressure is valid.

II. C.7. (Continued)

c. Thus, we take $\underset{\approx}{P} = p\underset{\approx}{I} + \underset{\approx}{\Pi} \Rightarrow -\nabla\cdot\underset{\approx}{P} = -\nabla p - \nabla\cdot\underset{\approx}{\Pi}$

d. Recall from Lect #14 that $\mathcal{O}\left(\frac{|\underset{\approx}{\Pi}|}{p}\right) \sim \frac{\lambda_m}{L}$.

Thus, we can neglect $\underset{\approx}{\Pi}$ (viscosity) term for $\lambda_m \ll L$.

e. Therefore, the <u>Limit of Strong Collisions</u> $\tau \gg \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}}\frac{1}{\nu_{ei}}$ and $\lambda_m \ll L$ enables the approximation of isotropic pressure p and the neglect of viscosity $\underset{\approx}{\Pi}$.

8. Using Continuity Eq, the LHS of Momentum equation can be simplified, to obtain

$$\boxed{\rho\frac{\partial \underset{\sim}{U}}{\partial t} + \rho\underset{\sim}{U}\cdot\nabla\underset{\sim}{U} = -\nabla p + \rho_q\underset{\sim}{E} + \underset{\sim}{j}\times\underset{\sim}{B}}$$

D. <u>Ampere's Law</u>:

1. In the nonrelativistic limit, we find we can neglect the displacement current in the Ampere-Maxwell Law

$$\nabla\times\underset{\sim}{B} = \mu_0\underset{\sim}{J} + \mu_0\epsilon_0\frac{\partial E}{\partial t}$$

a. We take $\nabla \sim \frac{1}{L}$ and $\frac{\partial}{\partial t} \sim \frac{1}{T}$ when estimating order of magnitude.

b. Looking at the ratio of the order of magnitudes of

$$\frac{\left|\mu_0\epsilon_0\frac{\partial E}{\partial t}\right|}{|\nabla\times\underset{\sim}{B}|} \sim \left(\frac{1}{c^2}\right)\frac{\frac{E}{T}}{B/L} \sim \frac{1}{c^2}\underbrace{\frac{L}{T}}_{\sim V_0}\underbrace{\frac{E}{B}}_{E\sim U\times B\sim V_0 B} \sim \frac{V_0}{c^2}\frac{V_0\cancel{B}}{\cancel{B}} \sim \frac{V_0^2}{c^2} \ll 1$$

c. Thus, in the Non-relativistic Limit $\boxed{\frac{V_0^2}{c^2} \ll 1}$, we use

$\boxed{\nabla\times\underset{\sim}{B} = \mu_0\underset{\sim}{J}}$ <u>Ampere's Law</u>

II. (Continued)

E. Ohm's Law

1. Last time, we derived the Generalized Ohm's Law

$$\underset{\sim}{E} + \underset{\sim}{U} \times \underset{\sim}{B} = \eta \underset{\sim}{j} + \underbrace{\frac{\underset{\sim}{j} \times \underset{\sim}{B}}{e n_e} - \frac{\nabla p_e}{e n_e}}_{\text{Hall terms}}$$

2. We'll show the Hall terms are small in the limit $r_{Li} \ll L$.

a. $$\frac{\left|\frac{\nabla p_e}{e n_e}\right|}{\underset{\sim}{U} \times \underset{\sim}{B}} \sim \frac{n_e k T_e}{e n_e V_0 B L} \underset{T_e = T_i}{\sim} \left(\frac{m_i}{e B_0}\right) \frac{1}{2} \left(\frac{2 k T_i}{m_i}\right) \frac{1}{V_0 L} \sim \frac{v_{ti}}{\omega_{ci}} \underbrace{\left(\frac{v_{ti}}{2 V_0}\right)}_{\sim 1} \frac{1}{L} \sim \frac{r_{Li}}{L} \ll 1$$

b. $$\frac{\frac{\underset{\sim}{j} \times \underset{\sim}{B}}{e n_e}}{\underset{\sim}{U} \times \underset{\sim}{B}} \sim \frac{1}{\mu_0} \frac{\nabla \times \underset{\sim}{B} \times \underset{\sim}{B}}{e n_e \underset{\sim}{U} \times \underset{\sim}{B}} \sim \frac{1}{\mu_0} \frac{B^2}{L e n_e V_0 B} \sim \underbrace{\left(\frac{B_0^2}{2 \mu_0 n k T_i}\right)}_{\substack{\frac{1}{\beta} \\ \text{plasma beta}}} \underbrace{\left(\frac{2 k T_i}{m_i}\right)}_{v_{ti}^2} \underbrace{\left(\frac{m_i}{e B_0}\right)}_{\frac{1}{\omega_{ci}}} \frac{1}{V_0 L}$$

$$\sim \frac{1}{\beta} \frac{v_{ti}}{\omega_{ci}} \cancelto{1}{\frac{v_{ti}}{V_0}} \frac{1}{L} \sim \frac{1}{\beta} \frac{r_{Li}}{L} \ll 1$$

c. Thus, in the $\boxed{\text{Magnetized Limit } r_{Li}/L \ll 1}$, Hall terms are negligible

3. Resistive Term:

a. $$\frac{\eta \underset{\sim}{j}}{\underset{\sim}{U} \times \underset{\sim}{B}} \sim \frac{\eta \frac{1}{\mu_0} \nabla \times B}{\underset{\sim}{U} \times \underset{\sim}{B}} \sim \left(\frac{m_e \nu_{ei}}{e^2 n_0}\right) \frac{1}{\mu_0} \frac{B}{L V_0 B} \sim \underbrace{\left(\frac{B^2}{2 \mu_0 n_0 k T_i}\right)}_{\frac{1}{\beta}} \left(\frac{\nu_{ei}}{L V_0}\right) \underbrace{\left(\frac{2 k T_i}{m_i}\right)}_{v_{ti}^2} \underbrace{\left(\frac{m_i^2}{e^2 B^2}\right)}_{\frac{1}{\omega_{ci}^2}} \left(\frac{m_e}{m_i}\right)$$

$$\sim \frac{1}{\beta} \underbrace{\left(\frac{\nu_{ei}}{L V_0}\right)}_{= \frac{\nu_{ei} \tau}{L^2}} \frac{v_{ti}^2}{\omega_{ci}^2} \frac{m_e}{m_i} \sim \frac{1}{\beta} \underbrace{(\tau \nu_{ei})}_{\text{large}} \underbrace{\left(\frac{r_{Li}}{L}\right)^2}_{\text{small}} \underbrace{\left(\frac{m_e}{m_i}\right)}_{\text{small}} \ll 1$$

b. The resistive term is often very small for plasmas.

$\Rightarrow$ This leads to the limit of IDEAL MHD when $\eta = 0$

4. In General $\boxed{\underset{\sim}{E} + \underset{\sim}{U} \times \underset{\sim}{B} = \eta \underset{\sim}{j}}$, but $\eta \underset{\sim}{j}$ is small so $\underset{\sim}{E} \simeq -\underset{\sim}{U} \times \underset{\sim}{B}$

II. E. (Continued)

5. NOTE: $\underset{\sim}{E} \simeq -\underset{\sim}{U} \times \underset{\sim}{B}$ is just the fluid manifestation of the $\underset{\sim}{E} \times \underset{\sim}{B}$ drift.

$\boxed{\text{Thus, the motion in a MHD fluid, given by the fluid velocity } \underset{\sim}{U}, \text{ is, to lowest order, just the } \underset{\sim}{E} \times \underset{\sim}{B} \text{ drift velocity.}}$

a. In determining order of magnitude estimates, we may take $\mathcal{O}(\underset{\sim}{E}) \sim \mathcal{O}(\underset{\sim}{U} \times \underset{\sim}{B})$.

F. Further Simplification of Momentum Equation

1. With the preceding simplifications, we can now estimate the size of the $\rho_q \underset{\sim}{E}$ term relative to $\underset{\sim}{j} \times \underset{\sim}{B}$ term.

$$\frac{\rho_q \underset{\sim}{E}}{\underset{\sim}{j} \times \underset{\sim}{B}} \sim \frac{\varepsilon_0 (\nabla \cdot \underset{\sim}{E}) \underset{\sim}{E}}{\frac{1}{\mu_0}(\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B}} \sim \frac{\underbrace{\mu_0 \varepsilon_0}_{=\frac{1}{c^2}} E^2 \cancel{L}}{\cancel{L} B^2} \overset{E \sim V_0 B}{\sim} \left(\frac{1}{c^2}\right) \frac{V_0^2 \cancel{B^2}}{\cancel{B^2}} \sim \frac{V_0^2}{c^2} \ll 1$$

2. In Non-relativistic Limit, $\frac{V_0^2}{c^2} \ll 1$, we can drop $\rho_q \underset{\sim}{E}$ term.

$$\boxed{\rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho \underset{\sim}{U} \cdot \nabla \underset{\sim}{U} = -\nabla p + \underset{\sim}{j} \times \underset{\sim}{B}}$$

3. Quasineutrality:

For MHD fluids, $\rho_q = \sum_s n_s q_s \simeq 0$.

a. Charge density: $$\frac{n_i - n_e}{n_e} \sim \frac{\overbrace{e(n_i - n_e)}^{=\rho_q}}{e n_e} \sim \frac{\varepsilon_0 \nabla \cdot \underset{\sim}{E}}{e n_e} \overset{\varepsilon_0 = \frac{1}{\mu_0 c^2}}{\sim} \frac{1}{c^2 \mu_0 e n_0} \frac{V_0 B}{L}$$

$$\sim \underbrace{\left(\frac{B^2}{2\mu_0 n k T_i}\right)}_{=\frac{1}{\beta}} \frac{1}{c^2} \frac{V_0}{L} \underbrace{\left(\frac{2kT_i}{m_i}\right)}_{v_{ti}^2} \underbrace{\left(\frac{m_i}{eB}\right)}_{\frac{1}{\Omega_i}} \sim \frac{1}{\beta} \frac{V_0 v_{ti}}{c^2} \frac{r_{Li}}{L} \sim \frac{1}{\beta} \underbrace{\left(\frac{V^2}{c^2}\right)}_{\text{small}} \underbrace{\left(\frac{r_{Li}}{L}\right)}_{\text{small}} \ll 1$$

b. Thus, MHD fluids maintain $\boxed{\text{Quasineutrality } \sum_s n_s q_s \simeq 0}$

II. (Continued)

G. Equation of State:

1. We choose to adopt the Adiabatic Equation of State from Two Fluid Theory,

$$\boxed{\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right)=0} \quad \text{with } \gamma = \tfrac{5}{3}$$

III. Summary of the MHD Equations

A. System of Equations:

1. Continuity Equation $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underset{\sim}{U}) = 0$ (Evolves ρ)

2. Momentum Equation $\rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho \underset{\sim}{U}\cdot\nabla \underset{\sim}{U} = -\nabla p + \underset{\sim}{j}\times \underset{\sim}{B}$ (Evolves $\underset{\sim}{U}$)

3. Adiabatic Equation of State (Energy Equation) $\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right)=0$ (Evolves p)

4. Faraday's Law $\frac{\partial \underset{\sim}{B}}{\partial t} = -\nabla\times \underset{\sim}{E}$ (Evolves $\underset{\sim}{B}$)

5. Ohm's Law $\underset{\sim}{E} + \underset{\sim}{U}\times \underset{\sim}{B} = \eta \underset{\sim}{j}$ (Determines $\underset{\sim}{E}$)

6. Ampere's Law $\nabla\times \underset{\sim}{B} = \mu_0 \underset{\sim}{j}$ (Determines $\underset{\sim}{J}$)

7. Zero Magnetic Divergence $\nabla\cdot \underset{\sim}{B} = 0$ (Constraint on $\underset{\sim}{B}$)

8. Gauss's Law $\nabla\cdot \underset{\sim}{E} = 0$ (Follows from $\nabla\cdot$(Ampere's Law))

9. NOTE: a. These are the equations of Resistive MHD

b. In the limit $\eta\to 0$, we have Ideal MHD.

III. (Continued)

B. The MHD Approximation: MHD Equations are Valid when:

1. Strong Collisions: $\tau \gg \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \nu_{ei}^{-1}$

$\lambda_m \ll L$

2. Non-relativistic: $V_0^2/c^2 \ll 1$
3. Magnetized: $r_{Li} \ll L$
4. Quasineutrality: $\sum_s n_s q_s \simeq 0$.

C. Simplified MHD Equations:

We can eliminate $\underset{\sim}{J}$ and $\underset{\sim}{E}$ to yield.

1. $$\frac{\partial \underset{\sim}{B}}{\partial t} = -\nabla \times \underset{\sim}{E} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B}) - \nabla \times \eta \underset{\sim}{j} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B}) - \frac{\eta}{\mu_0} \nabla \times (\nabla \times \underset{\sim}{B})$$

$$\boxed{\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B}) + \frac{\eta}{\mu_0} \nabla^2 \underset{\sim}{B}}$$

2. $$\underset{\sim}{j} \times \underset{\sim}{B} = \frac{1}{\mu_0}(\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B} \underset{\text{See Lect \#1}}{=} -\nabla\left(\frac{B^2}{2\mu_0}\right) + \frac{(\underset{\sim}{B} \cdot \nabla)\underset{\sim}{B}}{\mu_0}$$

Thus

$$\boxed{\rho \frac{d\underset{\sim}{U}}{\partial t} + \rho \underset{\sim}{U} \cdot \nabla \underset{\sim}{U} = -\nabla\Big(p + \underbrace{\frac{B^2}{2\mu_0}}_{\text{Magnetic Pressure}}\Big) + \underbrace{\frac{(\underset{\sim}{B} \cdot \nabla)\underset{\sim}{B}}{\mu_0}}_{\text{Magnetic Tension}}}$$

3. Therefore, we get the simplified set of equations

Resistive MHD Equations

Continuity Eq.	$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{U}) = 0$
Momentum Eq.	$\rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho \underset{\sim}{U} \cdot \nabla \underset{\sim}{U} = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{(\underset{\sim}{B} \cdot \nabla)\underset{\sim}{B}}{\mu_0}$
Induction Eq.	$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B}) + \frac{\eta}{\mu_0} \nabla^2 \underset{\sim}{B}$
Energy Eq.	$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0$

This is a closed set of 8 equations for 8 unknowns: $\rho, \underset{\sim}{U}, \underset{\sim}{B}, p$