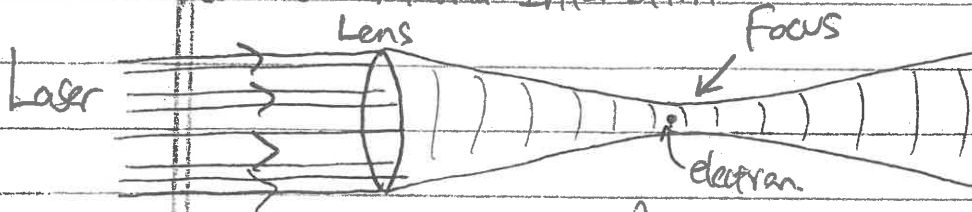


Lecture #10 The Ponderomotive Force

Howes ①

I. Particle Motion in High Frequency Electromagnetic Wave

A. Laser Plasma Interaction:



1. What is the motion of an electron in a high-frequency electromagnetic wave with variation in wave amplitude over space, i.e., near the focal point of a laser?

2. In this case, the plasma is unmagnetized. Only \underline{E} and \underline{B} from wave are present.

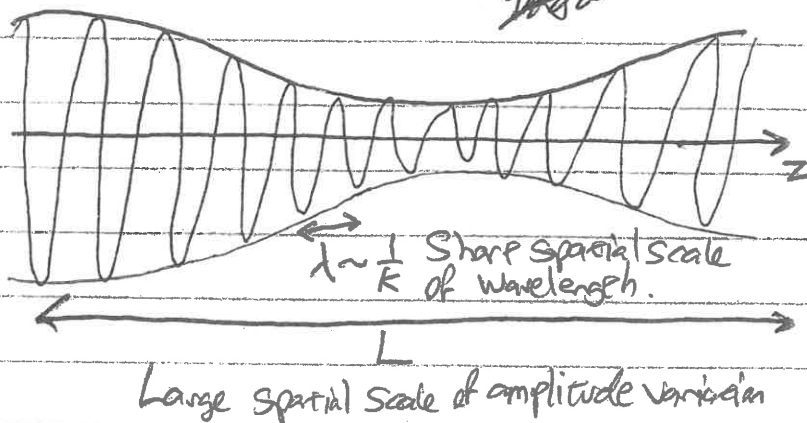
3. We'll use multiple timescale analysis to determine the lowest order nonlinear effect.

B. Multiple Timescale Analysis:

1. Consider an electromagnetic wave of high frequency ω whose amplitude may vary on a long timescale and large spatial scale.

$$\underline{E}(x,t) = \underline{E}_0(x,t) \cos(\omega t - \underline{k} \cdot \underline{x})$$

a. Two spatial scales:

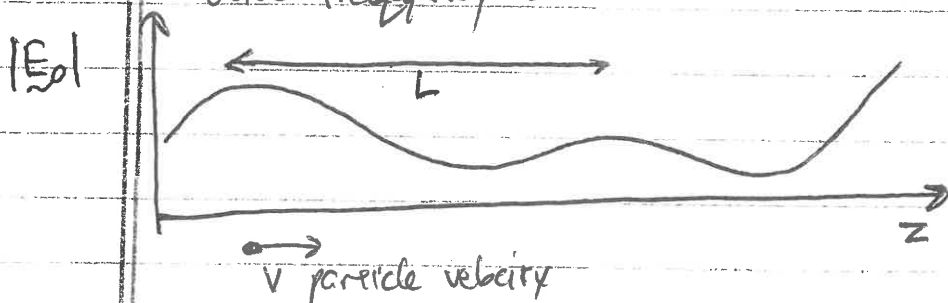


Z.B. (Continued)

2. We'll show in HW#5 that $\underline{E}(\underline{x}, T, t)$ yields

$$\underline{B}(\underline{x}, T, t) = -\frac{1}{\omega} \left\{ \nabla \times \underline{E}_0(\underline{k}, T) \sin(\omega t - \underline{k} \cdot \underline{x}) - \underline{k} \times \underline{E}_0(\underline{k}, T) \cos(\omega t - \underline{k} \cdot \underline{x}) \right\}$$

3. We want a ~~slow~~ slow variation of EM wave magnitude due to motion of the particle through space compared to the wave frequency ω !



a. To particle, amplitude varies in time due to motion $\underline{v} \cdot \nabla E_0$

$$\nabla \sim \frac{1}{L} \text{ large spatial scale} \quad |\underline{v} \cdot \nabla E_0| \sim \frac{v}{L} E_0$$

b. Frequency of EM wave gives $|\omega E_0| \sim \omega E_0$

c. We want $\frac{\text{Slow timescale}}{|\underline{v} \cdot \nabla E_0|} \ll \frac{\text{Fast timescale}}{|\omega E_0|} \Rightarrow \frac{v}{L} \ll \omega$ or $\boxed{\frac{v}{L\omega} \ll 1}$

d. This will be our ordering parameter

$$\boxed{\epsilon \sim \frac{v}{L\omega} \ll 1}$$

This separates fast oscillation timescale due to EM wave from slow drift timescale due to amplitude variation

4. $\frac{d\underline{v}}{dt} = \frac{q}{m} (\underline{E} + \underline{v} \times \underline{B})$

Compare these terms: $\frac{|\underline{v} \times \underline{B}|}{|\underline{E}|} \sim \frac{v \|\underline{B}\| \sin \theta}{|\underline{E}|} \sim \frac{v \left(\frac{E_0}{L\omega} \right)}{E_0} \sim \frac{v}{L\omega} \ll 1$

$$\frac{\partial \underline{B}}{\partial t} = -\nabla \times \underline{E} \Rightarrow \omega \underline{B} \sim \frac{E_0}{L} \text{ or } \underline{B} \sim \frac{E_0}{L\omega}$$

Lecture 10 (Continued)

Howes ③

Z.B. (Continued)

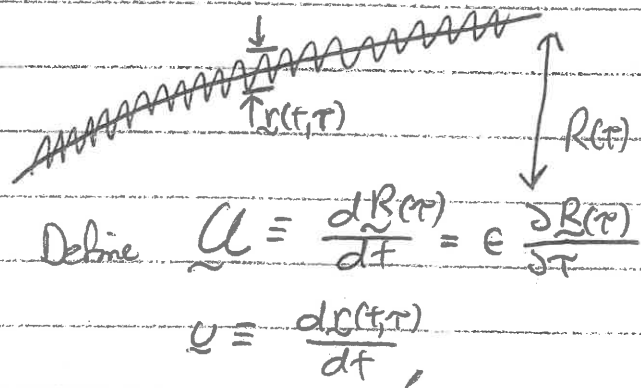
5. Two Timescales: a. ωt Fast oscillation timescale
 b. $\tau = \epsilon t$ Slow timescale of amplitude variation

c. Thus
$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau}$$

d. Over each fast oscillation period, ϵ varies little in amplitude. But small changes each oscillation can sum to produce a long timescale change.

6. Write particle position as slowly varying oscillation center $\underline{R}(\tau)$ plus small, rapidly oscillating position $\underline{r}(t, \tau)$

$$\underline{x} = \underline{R}(\tau) + \epsilon \underline{r}(t, \tau)$$



7. Velocity:

$$\underline{v} = \frac{d\underline{x}}{dt} = \frac{d\underline{R}(\tau)}{dt} + \epsilon \frac{d\underline{r}(t, \tau)}{dt}$$

Define $\underline{u} = \frac{d\underline{R}(\tau)}{d\tau} = \epsilon \frac{\partial \underline{R}(\tau)}{\partial \tau}$

$$\underline{v} = \frac{d\underline{r}(t, \tau)}{dt}$$

$$\underline{v} = \epsilon \underline{u}(\tau) + \epsilon \underline{v}(t, \tau)$$

8. Acceleration:

$$\frac{d\underline{v}}{dt} = \epsilon \frac{d\underline{u}(\tau)}{dt} + \epsilon \frac{d\underline{v}(t, \tau)}{dt} = \epsilon \frac{\partial \underline{u}(\tau)}{\partial \tau} + \epsilon \frac{\partial \underline{u}(\tau)}{\partial \tau} + \epsilon \frac{\partial \underline{v}(t, \tau)}{\partial t} + \epsilon^2 \frac{\partial \underline{v}(t, \tau)}{\partial \tau}$$

9. Thus, we find:

$$\epsilon \frac{\partial \underline{v}(t, \tau)}{\partial t} + \epsilon^2 \frac{\partial \underline{u}(\tau)}{\partial \tau} + \epsilon^2 \frac{\partial \underline{v}(t, \tau)}{\partial \tau} = \frac{q}{m} \left[\underline{E}(\underline{R}(t, \tau), t, \tau) + (\epsilon \underline{u}(\tau) + \epsilon \underline{v}(t, \tau)) \times \underline{B}(\underline{R}(t, \tau), t, \tau) \right]$$

a NOTE that highest order nonzero term of LHS is $\mathcal{O}(\epsilon)$.

Thus, highest term on RHS must be $\mathcal{O}(\epsilon)$ to balance. Hence, we multiply RHS by ϵ to give balance

Lecture #10 (Continued)

Homes 4

Z.B. (Continued)

0. Taylor Expand fields about oscillation center \underline{R} :

$$a. \underline{E}(\underline{x}, \tau, t) = \underline{E}(\underline{R}, \tau, t) + (\underline{x} - \underline{R}) \cdot \nabla \underline{E}(\underline{R}, \tau, t) + \frac{[(\underline{x} - \underline{R}) \cdot \nabla]^2}{2!} \underline{E}(\underline{R}, \tau, t) + \dots$$

b. NOTE: $\underline{x} - \underline{R} = \underline{r}$, so

$$\underline{E}(\underline{x}, \tau, t) = \underline{E}(\underline{R}, \tau, t) + \epsilon \underline{r} \cdot \nabla \underline{E}(\underline{R}, \tau, t) + \dots$$

and likewise with $\underline{B}(\underline{x}, \tau, t)$

1. Expand all variables and substitute:

$$a. \underline{U}(\tau) = \underline{U}_1(\tau) + \epsilon \underline{U}_2(\tau) + \dots$$

$$\underline{U}(\tau, t) = \underline{U}_1(\tau, t) + \epsilon \underline{U}_2(\tau, t) + \dots$$

$$\underline{r}(\tau, t) = \underline{r}_1(\tau, t) + \epsilon \underline{r}_2(\tau, t) + \dots$$

2. Thus, we get

$$\begin{aligned} & \epsilon \frac{\partial \underline{U}_1(\tau, t)}{\partial t} + \epsilon^2 \frac{\partial \underline{U}_2(\tau, t)}{\partial t} + \epsilon^2 \frac{\partial \underline{U}_1(\tau)}{\partial \tau} + \epsilon^3 \frac{\partial \underline{U}_2(\tau)}{\partial \tau} + \epsilon^2 \frac{\partial \underline{U}_1(\tau, t)}{\partial \tau} + \epsilon^3 \frac{\partial \underline{U}_2(\tau, t)}{\partial \tau} \\ & = \frac{q}{m} \left[\epsilon \underline{E}(\underline{R}, \tau, t) + \epsilon^2 \underline{r}_1 \cdot \nabla \underline{E}(\underline{R}, \tau, t) + \epsilon^2 \underline{U}_1 \times \underline{B}(\underline{R}, \tau, t) + \epsilon^3 \underline{U}_2 \times \underline{B}(\underline{R}, \tau, t) \right. \\ & \quad + \epsilon^3 \underline{U}_1 \times (\underline{r}_1 \cdot \nabla) \underline{B}(\underline{R}, \tau, t) + \epsilon^4 \underline{U}_2 \times (\underline{r}_1 \cdot \nabla) \underline{B}(\underline{R}, \tau, t) + \epsilon^2 \underline{U}_1 \times \underline{B}(\underline{R}, \tau, t) \\ & \quad \left. + \epsilon^3 \underline{U}_2 \times \underline{B}(\underline{R}, \tau, t) + \epsilon^3 \underline{U}_1 \times (\underline{r}_1 \cdot \nabla) \underline{B}(\underline{R}, \tau, t) + \epsilon^4 \underline{U}_2 \times (\underline{r}_1 \cdot \nabla) \underline{B}(\underline{R}, \tau, t) + \dots \right] \end{aligned}$$

3. Lowest Order: $\mathcal{O}(\epsilon)$

$$\frac{\partial \underline{U}_1(\tau, t)}{\partial t} = \frac{q}{m} \underline{E}_0(\underline{R}, \tau) \cos(\omega t - \underline{k} \cdot \underline{R})$$

$\frac{d\underline{p}}{dt}$

$$= \underline{U}_1(\tau, t) = \frac{q}{m\omega} \underline{E}_0(\underline{R}, \tau) \sin(\omega t - \underline{k} \cdot \underline{R}) \leftarrow \text{Oscillation Velocity}$$

$$\underline{r}_1(\tau, t) = \frac{-q}{m\omega^2} \underline{E}_0(\underline{R}, \tau) \cos(\omega t - \underline{k} \cdot \underline{R}) \leftarrow \text{Oscillation position}$$

I. B. (Continued)

14. ~~Next~~ Order: $\mathcal{O}(e^2)$

$$a. \frac{\partial u_2(t, \underline{r})}{\partial t} \textcircled{1} + \frac{\partial u_1(t, \underline{r})}{\partial t} \textcircled{2} + \frac{\partial u(t, \underline{r})}{\partial t} \textcircled{3} = \frac{q}{m} \left[\underline{n}_1 \cdot \nabla \underline{E}(\underline{r}, t) \textcircled{4} + \underline{u}_1 \times \underline{B}(\underline{r}, t) \textcircled{5} + \underline{u} \times \underline{B}(\underline{r}, t) \textcircled{6} \right]$$

b. We now average over oscillation period to annihilate terms $\frac{\omega}{2\pi} \int_0^{2\pi} dt$.

i) Assume $u_2(t, \underline{r})$ is periodic over $T = \frac{2\pi}{\omega}$.

This should be checked a posteriori.

c. Term ①: $\frac{\omega}{2\pi} \int_0^{2\pi} \frac{\partial u_2(t, \underline{r})}{\partial t} dt = 0$ by assumed periodicity

d. Term ②: $\frac{\omega}{2\pi} \int_0^{2\pi} \frac{\partial u_1(t, \underline{r})}{\partial t} dt = \frac{\partial u_1(t, \underline{r})}{\partial t} \frac{\omega}{2\pi} \int_0^{2\pi} dt = \frac{\partial u_1(t, \underline{r})}{\partial t}$

e. Term ③: $\frac{\omega}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial t} \left[\frac{q}{m\omega} E_0(\underline{r}, t) \sin(\omega t - \underline{k} \cdot \underline{r}) \right] dt$
 $= \frac{q}{m\omega} \frac{\partial E_0(\underline{r}, t)}{\partial t} \frac{\omega}{2\pi} \int_0^{2\pi} \sin(\omega t - \underline{k} \cdot \underline{r}) dt = 0$

f. Term ④: $\frac{\omega}{2\pi} \int_0^{2\pi} \frac{q}{m} \left[\frac{-q}{m\omega^2} E_0(\underline{r}, t) \cos(\omega t - \underline{k} \cdot \underline{r}) \right] \cdot \nabla E_0(\underline{r}, t) \cos(\omega t - \underline{k} \cdot \underline{r}) dt$
 $= \frac{-q^2}{m\omega^2} E_0(\underline{r}, t) \cdot \nabla E_0(\underline{r}, t) \frac{\omega}{2\pi} \int_0^{2\pi} \underbrace{\cos^2(\omega t - \underline{k} \cdot \underline{r})}_{=\frac{1}{2}} dt = \frac{-q^2}{2m^2\omega^2} E_0(\underline{r}, t) \cdot \nabla E_0(\underline{r}, t)$

g. Term ⑤: $\frac{\omega}{2\pi} \int_0^{2\pi} \frac{q}{m} \underline{u}_1(t, \underline{r}) \times \left[\frac{-1}{\omega} \left\{ \nabla \times E_0(\underline{r}, t) \sin(\omega t - \underline{k} \cdot \underline{r}) - \underline{k} \times E_0(\underline{r}, t) \cos(\omega t - \underline{k} \cdot \underline{r}) \right\} \right] dt$
 $= \frac{-q}{m\omega} \left[\underline{u}_1(t, \underline{r}) \times \nabla \times E_0(\underline{r}, t) \frac{\omega}{2\pi} \int_0^{2\pi} \sin(\omega t - \underline{k} \cdot \underline{r}) dt - \underline{u}_1(t, \underline{r}) \times \underline{k} \times E_0(\underline{r}, t) \frac{\omega}{2\pi} \int_0^{2\pi} \cos(\omega t - \underline{k} \cdot \underline{r}) dt \right]$
 $= 0$

Lecture #10 (Continued)

Pages 6

I. B.H. (Continued)

$$h. \text{ Term (6): } \frac{q}{2\pi} \int_0^{2\pi} \frac{q}{m} \left[\frac{q}{m\omega} \underline{E}_0(\underline{R}, \tau) \sin(\omega t - \underline{k} \cdot \underline{R}) \right] \times \left\{ \frac{-1}{\omega} \left[\nabla \times \underline{E}_0(\underline{R}, \tau) \sin(\omega t - \underline{k} \cdot \underline{R}) \right. \right. \\ \left. \left. - \underline{k} \times \underline{E}_0(\underline{R}, \tau) \cos(\omega t - \underline{k} \cdot \underline{R}) \right] \right\} dt$$

$$= -\frac{q^2}{m^2 \omega^2} \left\{ \underline{E}_0(\underline{R}, \tau) \times \nabla \times \underline{E}_0(\underline{R}, \tau) \frac{\omega}{2\pi} \int_0^{2\pi} \sin^2(\omega t - \underline{k} \cdot \underline{R}) dt \right. \\ \left. - \underline{E}_0(\underline{R}, \tau) \times \underline{k} \times \underline{E}_0(\underline{R}, \tau) \frac{\omega}{2\pi} \int_0^{2\pi} \sin(\omega t - \underline{k} \cdot \underline{R}) \cos(\omega t - \underline{k} \cdot \underline{R}) dt \right\}$$

$$= -\frac{q^2}{2m^2 \omega^2} \underline{E}_0(\underline{R}, \tau) \times \nabla \times \underline{E}_0(\underline{R}, \tau)$$

i. Putting Solution together, we find

$$\frac{\partial \underline{U}_1(\tau)}{\partial \tau} = -\frac{q^2}{2m^2 \omega^2} \left[\underline{E}_0(\underline{R}, \tau) \cdot \nabla \underline{E}_0(\underline{R}, \tau) + \underline{E}_0(\underline{R}, \tau) \times \nabla \times \underline{E}_0(\underline{R}, \tau) \right]$$

j. NOTE: NRL p. 4 (12) gives

$$\nabla(\underline{A} \cdot \underline{B}) = \underline{A} \times (\nabla \times \underline{B}) + \underline{B} \times (\nabla \times \underline{A}) + (\underline{A} \cdot \nabla) \underline{B} + (\underline{B} \cdot \nabla) \underline{A}$$

Taking $\underline{A} = \underline{B} = \underline{E}$, we get $\nabla \left(\frac{|\underline{E}|^2}{2} \right) = \underline{E} \times (\nabla \times \underline{E}) + (\underline{E} \cdot \nabla) \underline{E}$

k. Thus, we find:

$$\boxed{\frac{\partial \underline{U}_1(\tau)}{\partial \tau} = -\frac{q^2}{2m^2 \omega^2} \nabla \left(\frac{|\underline{E}_0(\underline{R}, \tau)|^2}{2} \right)} \quad \text{Ponderomotive Force}$$

I (Continued)

C. Properties of the Ponderomotive Force

1.
$$\underline{F}_{\text{pond}} = m \frac{d\underline{U}}{dt} = \frac{-q^2}{4m\omega^2} \nabla |\underline{E}_0|^2$$
 Pushes away from regions of intense field.

a. We can write this as a potential force

$$\underline{F}_{\text{pond}} = -\nabla \Phi_{\text{pond}}$$

where
$$\Phi_{\text{pond}} = \frac{q^2}{4m\omega^2} |\underline{E}_0|^2$$

b. Note that the average ~~base~~ oscillation ~~velocity~~ energy is

$$\begin{aligned} \frac{1}{2} m \overline{|\underline{U}|^2} &= \frac{m}{2} \frac{\omega}{2\pi} \int_0^{2\pi} \frac{q^2}{m^2 \omega^2} |\underline{E}_0|^2 \sin^2(\omega t - \underline{k} \cdot \underline{R}) dt = \frac{m}{2} \frac{\omega}{2\pi} \frac{q^2}{m^2 \omega^2} |\underline{E}_0|^2 \frac{2\pi}{2} \\ &= \frac{q^2}{4m\omega^2} |\underline{E}_0|^2 \end{aligned}$$

c. Thus
$$\Phi_{\text{pond}} = \frac{1}{2} m \overline{|\underline{U}|^2}$$

Ponderomotive potential is the average base oscillation kinetic energy.

d. For carrier of oscillation
$$\Sigma_{\text{osc}} = \frac{1}{2} m \underline{U}^2 + \Phi_{\text{pond}} = \frac{1}{2} m \underline{U}^2 + \frac{1}{2} m \overline{|\underline{U}|^2}$$

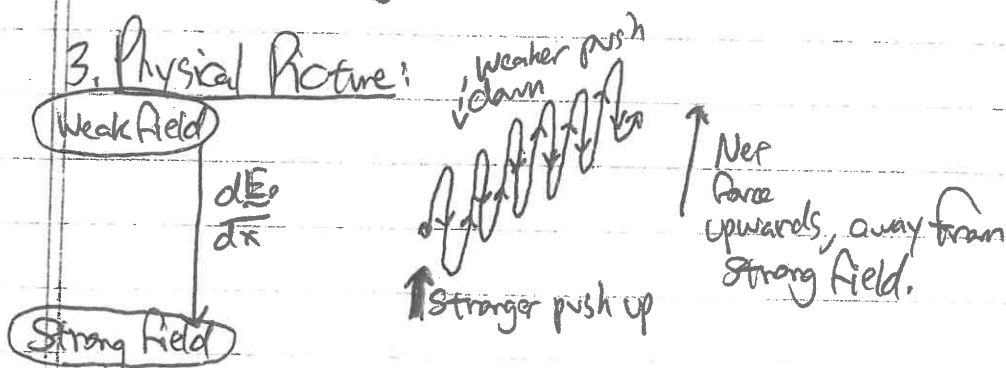
2. a. Force is independent of ~~sign~~ of charge

⇒ Repels both ions and electrons from high field regions.

b. Because $m_e \ll m_i$, electrons are pushed aside much more easily.

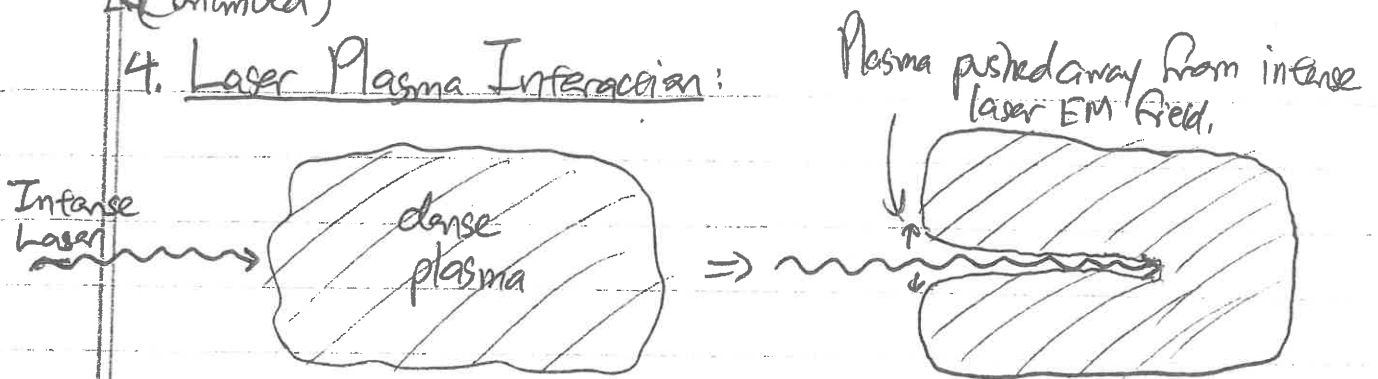
⇒ Resulting polarization electric field acts to pull ions out.

3. Physical Picture:



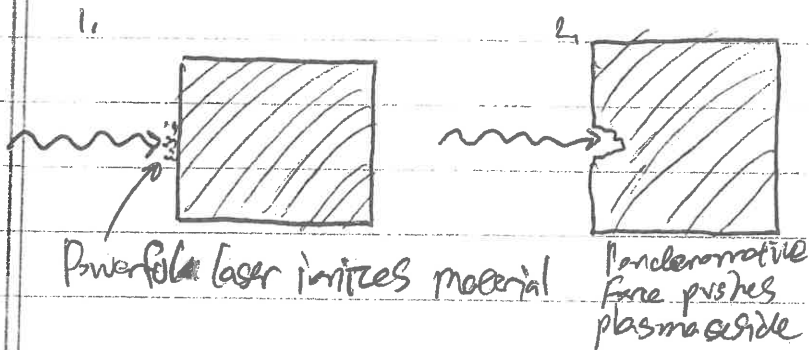
1 (Continued)

4. Laser Plasma Interaction:



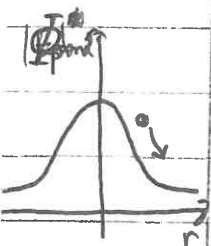
- a. This can lead to self-focusing of laser light in a plasma.
1. Powerful laser pushes aside electrons (and ions) due to ponderomotive force.
 2. The resulting depression in plasma density acts as a convex lens, focusing the laser light into the evacuated channel.

b. Lasers can bore holes in materials by this mechanism.



3. Laser can propagate if ~~the~~ light frequency $\omega > \omega_{pe}$ in plasma.

5. Example: A particle of charge q & mass m is initially at rest at the center of a Gaussian laser beam with $|E_0(x)| = E_0 e^{-\frac{r^2}{r_0^2}}$. Find Center of oscillation velocity as a function of position.



$$\Sigma_0 = \frac{1}{2} m v^2 + \Phi_{pond} \quad \text{where} \quad \Phi_{pond} = \frac{q^2}{4\pi m \epsilon_0^2} |E_0|^2 = \frac{q^2}{4\pi m \epsilon_0^2} E_0^2 e^{-\frac{2r^2}{r_0^2}}$$

At $r=0$, $\Sigma_0 = \frac{1}{2} m v^2 + \frac{q^2}{4\pi m \epsilon_0^2} E_0^2$

Thus $v = \sqrt{\frac{2\Sigma_0 - 2\Phi_{pond}}{m}} = \sqrt{\frac{2\Sigma_0}{m} - \frac{2q^2}{4\pi m \epsilon_0^2} e^{-\frac{2r^2}{r_0^2}}} = \sqrt{\frac{2\Sigma_0}{m} \left(1 - e^{-\frac{2r^2}{r_0^2}}\right)^{\frac{1}{2}}}$

$$v(r) = \frac{q E_0}{m \omega \sqrt{2}} \left(1 - e^{-\frac{2r^2}{r_0^2}}\right)^{\frac{1}{2}}$$