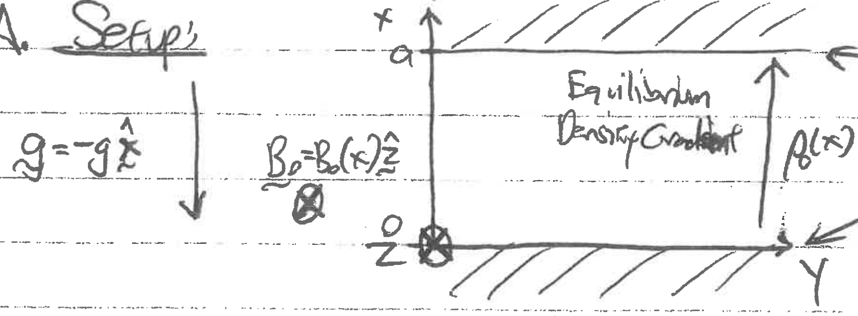
Lecture #13: MHD Stability Analysis of Rayleigh–Taylor Instability

I. Normal Mode Analysis

A. Setup:



Rigid, conducting walls at $x=0$, $x=a$.

Boundary Conditions:

$U_x(0) = U_x(a) = 0.$

1. Density has exponential form in direction of gravity ($\hat{x}$)

a. $\rho_0(x) = \rho_{00}\, e^{-\frac{x}{H}}$ $\quad H \equiv$ Scale height of density

b. For $H>0$, density decreases with height (stable)
$H<0$, density increases with height (unstable)

2. a. For simplicity, we assume $\frac{\partial}{\partial z} = 0$ (No variation along mean field).
NOTE: Such variations would bend the magnetic field lines, leading to magnetic tension (which stabilizes instability)
b. We also assume incompressible motion, $\nabla \cdot \underset{\sim}{U}_1 = 0$

3. We want to analyze this problem ~~for~~ stability.
a. Normal Mode Analysis
b. Energy Principle

B. Using Linear Force Operator

1. We could use
$$-\rho_0 \omega^2 \underset{\sim}{\xi} = \underset{\sim}{F}(\underset{\sim}{\xi})$$
to solve for the characteristic frequencies. But instead, we'll begin from equation of motion.

Lecture #13 (Continued)

I.B. Using Equation of Motion:

1. Momentum Eq:

$$\rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho\, \underset{\sim}{U}\cdot\nabla\underset{\sim}{U} = -\nabla\left(p+\frac{B^2}{2\mu_0}\right) + \frac{(\underset{\sim}{B}\cdot\nabla)\underset{\sim}{B}}{\mu_0} + \underbrace{\rho g}_{}$$

gravity, where $g = -g\hat{\underset{\sim}{x}}$

2. Linearize about Equilibrium:

$$\rho = \rho_0(x) + \epsilon\, \rho_1(\underset{\sim}{x})$$

$$\underset{\sim}{U} = \cancelto{0}{\underset{\sim}{U_0}(x)} + \epsilon\, \underset{\sim}{U_1}(\underset{\sim}{x})$$

$$\underset{\sim}{B} = B_0(x)\,\hat{\underset{\sim}{z}} + \epsilon\, \underset{\sim}{B_1}(\underset{\sim}{x})$$

$$p = \underset{\substack{\uparrow\\ \text{vertical}\\ \text{direction}\\ \text{(scalar)}}}{p_0(x)} + \epsilon\, p_1(\underset{\substack{\uparrow\\ \text{vector}}}{\underset{\sim}{x}})$$

3. Lowest Order: $\mathcal{O}(1)$: $0 = -\nabla\left(p_0 + \frac{B_0^2}{2\mu_0}\right) + \rho_0 g$

a. Equilibrium satisfies: $\boxed{\frac{\partial}{\partial x}\left(p_0 + \frac{B_0^2}{2\mu_0}\right) = -\rho_0 g}$ Static Equilibrium

b. Notation: $p_0' = \frac{\partial p_0}{\partial x}$, $B_0' = \frac{\partial B_0}{\partial x}$, $\rho_0' = \frac{\partial \rho_0}{\partial x}$

$$\Rightarrow \boxed{p_0' + \frac{B_0 B_0'}{\mu_0} = -\rho_0 g}$$

4. Next Order: $\mathcal{O}(\epsilon)$:

$$\underset{\textcircled{1}}{\rho_0 \frac{\partial \underset{\sim}{U_1}}{\partial t}} = \underset{\textcircled{2}}{-\nabla\left(p_1 + \frac{\underset{\sim}{B_0}\cdot\underset{\sim}{B_1}}{\mu_0}\right)} + \underset{\textcircled{3}}{\frac{(\underset{\sim}{B_0}\cdot\nabla)\underset{\sim}{B_1}}{\mu_0}} + \underset{\textcircled{4}}{\frac{\underset{\sim}{B_1}\cdot\nabla\underset{\sim}{B_0}}{\mu_0}} + \underset{\textcircled{5}}{\rho_1 g}$$

a. Term ③ $(\underset{\sim}{B_0}\cdot\nabla)\underset{\sim}{B_1} = B_0 \cancelto{0}{\frac{\partial}{\partial z}}\underset{\sim}{B_1} = 0$

b. Term ④ $(\underset{\sim}{B_1}\cdot\nabla)\underset{\sim}{B_0} = B_x \frac{\partial B_0 \hat{\underset{\sim}{z}}}{\partial x} = B_0' B_x \hat{\underset{\sim}{z}}$

c. We can eliminate Term ② by taking the curl of this equation!

$(\nabla\times\nabla\underset{\substack{\uparrow\\ \text{scalar function.}}}{\phi} = 0)$.

Lecture #3 (Continued)

I. C. 4. (Continued)

d. $\nabla\times\left(\rho_0 \frac{\partial \underset{\sim}{U}_1}{\partial t}\right) = \nabla\times(B_0' B_x \hat{\underset{\sim}{z}}) + \nabla\times(-\rho_1 g \hat{\underset{\sim}{x}})$

e. NOTE: Since $\frac{\partial}{\partial z} = 0$, $\nabla = \frac{\partial}{\partial x}\hat{\underset{\sim}{x}} + \frac{\partial}{\partial y}\hat{\underset{\sim}{y}}$

f. Let's find the $\hat{\underset{\sim}{z}}$- component:

1. $\hat{\underset{\sim}{z}} \cdot \left[\nabla\times\left(\rho_0 \frac{\partial \underset{\sim}{U}_1}{\partial t}\right)\right] = \frac{\partial}{\partial x}\left[\rho_0 \frac{\partial U_y}{\partial t}\right] - \frac{\partial}{\partial y}\left[\rho_0 \frac{\partial U_x}{\partial t}\right]$

$$= \rho_0' \frac{\partial U_y}{\partial t} + \rho_0 \frac{\partial^2 U_y}{\partial x \partial t} - \rho_0 \frac{\partial^2 U_x}{\partial y \partial t}$$

2. $\hat{\underset{\sim}{z}} \cdot \left[\nabla\times(B_0' B_x \hat{\underset{\sim}{z}})\right] = 0$

3. $\hat{\underset{\sim}{z}} \cdot \left[\nabla\times(-\rho_1 g \hat{\underset{\sim}{x}})\right] = -\frac{\partial}{\partial y}\left[-\rho_1 g\right] = g\frac{\partial \rho_1}{\partial y}$

g. Thus, we find $\boxed{\rho_0' \frac{\partial U_y}{\partial t} + \rho_0 \frac{\partial^2 U_y}{\partial x \partial t} - \rho_0 \frac{\partial^2 U_x}{\partial y \partial t} = g\frac{\partial \rho_1}{\partial y}}$

5. Fourier Transform in y and t: $\underset{\sim}{U}_1(\underset{\sim}{x}) = \underset{\sim}{U}_1(x)\, e^{i(k_y y - \omega t)}$

a. Thus $\frac{\partial}{\partial t} = -i\omega$ $\frac{\partial}{\partial y} = ik_y$

b. This yields $\boxed{\rho_0' U_y + \rho_0 \frac{\partial U_y}{\partial x} - \rho_0 i k_y U_x = -\frac{k_y g}{\omega}\rho_1}$

6. We assume incompressible motion $\nabla\cdot\underset{\sim}{U}_1 = 0$

$\boxed{\frac{\partial U_x}{\partial x} + ik_y U_y = 0}$ $\Rightarrow U_y = \frac{i}{k_y}\frac{\partial U_x}{\partial x}$

7. Continuity Equation $\frac{\partial \rho}{\partial t} + \underset{\sim}{U}\cdot\nabla\rho + \rho\nabla\cdot U = 0$

a. $\mathcal{O}(\epsilon)$: $\frac{\partial \rho_1}{\partial t} + \underset{\sim}{U}_1\cdot\nabla\rho_0 + \cancelto{0}{\rho_0\nabla\cdot\underset{\sim}{U}_1} = 0$

b. $\boxed{-i\omega\rho_1 + U_x\rho_0' = 0}$ $\Rightarrow$ $\rho_1 = \frac{-i}{\omega}U_x\rho_0'$

Lecture #3 (Continued)

I.C. (Continued)

8. Eliminate U_y & ρ_1 in favor of U_x:

a. This yields:

$$\frac{\partial^2 U_x}{\partial x^2} + \frac{\rho_0'}{\rho_0}\frac{\partial U_x}{\partial x} - k_y^2\left[1 + \frac{g}{\omega^2}\left(\frac{\rho_0'}{\rho_0}\right)\right] U_x = 0$$

9. Note: since $\rho_0(x) = \rho_{00} e^{-\frac{x}{H}}$, we have $\frac{\rho_0'}{\rho_0} = -\frac{1}{H}$

$$\frac{\partial^2 U_x}{\partial x^2} - \frac{1}{H}\frac{\partial U_x}{\partial x} - k_y^2\left(1 - \frac{g}{\omega^2 H}\right) U_x = 0$$

10. We can solve this with the help of an integrating factor.

a. Let $U_x(x) = f(x)\, e^{+\frac{x}{2H}}$

b. This gives $\frac{\partial^2 U_x}{\partial x^2} - \frac{1}{H}\frac{\partial U_x}{\partial x} = \left(\frac{\partial^2 f}{\partial x^2} - \frac{f}{4H^2}\right) e^{+\frac{x}{2H}}$

c. Thus, we find:

$$\frac{\partial^2 f}{\partial x^2} + \alpha^2 f = 0 \quad \text{where} \quad \alpha^2 = k_y^2\left(\frac{g}{H\omega^2} - 1\right) - \frac{1}{4H^2}$$

11. The function $f(x)$ must satisfy the boundary conditions

$U_x(0) = U_x(a) = 0 \quad \Rightarrow f(0) = f(a) = 0$

a. General Solution: $f(x) = f_0 \sin\alpha x + f_1 \cos\alpha x$

b. $f(0) = f_0(0) + f_1(1) = 0 \Rightarrow f_1 = 0.$

c. $f(a) = f_0 \sin(\alpha a) = 0 \Rightarrow \alpha = \frac{n\pi}{a}$ for $n = 1, 2, 3, \ldots$

d. We therefor have eigenfunctions f_n with mode number n.

12. Solve for frequency: $\frac{n^2\pi^2}{a^2} = k_y^2\left(\frac{g}{H\omega_n^2} - 1\right) - \frac{1}{4H^2}$

a. $\frac{n^2\pi^2}{a^2} + k_y^2 + \frac{1}{4H^2} = \frac{g}{H}\frac{k_y^2}{\omega_n^2}$

Lecture #13 (Continued)

I. C.R. (Continued)

b. $$\boxed{\omega_n^2 = \left(\frac{g}{H}\right) \underbrace{\frac{4H^2 k_y^2 a^2}{a^2 + 4H^2(n^2\pi^2 + k_y^2 a^2)}}_{\text{positive definite}}}$$

c. For $H > 0$, $\omega_n^2 > 0 \Rightarrow$ Oscillating function $\Rightarrow$ STABLE

For $H < 0$, $\omega_n^2 < 0 \Rightarrow$ UNSTABLE.

d. 1. Max growth rate $\omega_n^2 = \frac{g}{H}$ as $k_y \to \infty$

2. Growth rate $\to 0$ as $k_y \to 0$

3. Lower vertical mode numbers n have faster growth.

II. Energy Principle:

A. Gravitational Force Term:

1. In Linear Force Operator $\underset{\sim}{F}(\underset{\sim}{\xi})$, we must add gravity term:

$$+\rho_1 \underset{\sim}{g}$$

a. From Lecture #1, II. A, 4, b, 2, we have $\rho_1 = -\underset{\sim}{\xi}\cdot\nabla\rho_0 - \rho_0 \nabla\cdot\underset{\sim}{\xi} = 0$

b. For incompressible motion, $\nabla\cdot\underset{\sim}{\xi} = 0$, so $\rho_1 = -\underset{\sim}{\xi}\cdot\nabla\rho_0 = -\xi_x \rho_0'$

c. Thus $+\rho_1 \underset{\sim}{g} = (-\xi_x \rho_0')(-g\hat{\underset{\sim}{x}}) = \rho_0' g \xi_x \hat{\underset{\sim}{x}}$

2. This gives:

$$\underset{\sim}{F}(\underset{\sim}{\xi}) = \nabla\left[\underset{\sim}{\xi}\cdot\nabla p_0 + \gamma p_0 \cancelto{0}{\nabla\cdot\underset{\sim}{\xi}}\right] + \frac{(\nabla\times\underset{\sim}{B}_0)\times[\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}_0)] + (\nabla\times[\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}_0)])\times\underset{\sim}{B}_0}{\mu_0}$$

$$+ \rho_0' g \xi_x \hat{\underset{\sim}{x}}$$

3. For the energy principle, we must add this term:

$$-\frac{1}{2}|\xi_x|^2 g \rho_0'$$

$$\left[\text{Remember, } \delta W = -\frac{1}{2}\int d^3\underset{\sim}{x}\, \underset{\sim}{\xi}\cdot\underset{\sim}{F}(\underset{\sim}{\xi})\right]$$

II. (Continued)

B. Using Energy Principle

1. With the added gravitational potential term, we have

$$\delta W = \frac{1}{2}\int d^3\mathbf{x}\left\{\underset{\textcircled{1}}{\frac{|\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)|^2}{\mu_0}} + \underset{\textcircled{2}}{\gamma p_0 |\cancelto{0}{\nabla\cdot\boldsymbol{\xi}}|^2} - \underset{\textcircled{3}}{\boldsymbol{\xi}^*\cdot\mathbf{j}_0\times[\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)]} - \underset{\textcircled{4}}{\boldsymbol{\xi}^*\cdot\nabla(\boldsymbol{\xi}\cdot\nabla p_0)} - \underset{\textcircled{5}}{|\xi_x|^2 g \rho_0'}\right\}$$

2. a. $\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0) \underset{\uparrow}{=} \boldsymbol{\xi}(\cancelto{0}{\nabla\cdot\mathbf{B}_0}) - \mathbf{B}_0(\cancelto{0}{\nabla\cdot\boldsymbol{\xi}}) + \underbrace{(\mathbf{B}_0\cdot\nabla)\boldsymbol{\xi}}_{\cancelto{0}{B_0\partial_z}} - (\boldsymbol{\xi}\cdot\nabla)\mathbf{B}_0$

NRL (10) p.4

$$= -B_0'\xi_x\hat{\mathbf{z}}$$

b. $\mathbf{j}_0 = \frac{\nabla\times\mathbf{B}_0}{\mu_0} = \frac{1}{\mu_0}\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}\right)\times(B_0(x)\hat{\mathbf{z}}) = \frac{-1}{\mu_0}B_0'\hat{\mathbf{y}}$

3. TERM ①: $= \frac{(B_0')^2}{\mu_0}|\xi_x|^2$

4. TERM ③: $= -\boldsymbol{\xi}^*\cdot\left(-\frac{B_0'}{\mu_0}\hat{\mathbf{y}}\right)\times(-B_0'\xi_x\hat{\mathbf{z}}) = -\boldsymbol{\xi}^*\cdot\left(\frac{(B_0')^2}{\mu_0}\xi_x\hat{x}\right) = -\frac{(B_0')^2}{\mu_0}|\xi_x|^2$

   a. Thus, Term ① + Term ③ = 0 ✓

5. Term ④: $-\boldsymbol{\xi}^*\cdot\nabla\underbrace{[(\boldsymbol{\xi}\cdot\nabla)p_0]}_{\text{Scalar} = f}$

   a. NOTE: NRL (7) p.4 $\nabla\cdot(f\mathbf{A}) = f\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla f$

   $\nabla\cdot(\boldsymbol{\xi}^* f) = f\,\cancelto{0}{\nabla\cdot\boldsymbol{\xi}^*} + \boldsymbol{\xi}^*\cdot\nabla f$

   b. Thus $\int d^3\mathbf{x}\left\{\text{[illegible]} - \boldsymbol{\xi}^*\cdot\nabla[(\boldsymbol{\xi}\cdot\nabla)p_0]\right\} = -\int d^3\mathbf{x}\,\nabla\cdot\left[\boldsymbol{\xi}^*(\boldsymbol{\xi}\cdot\nabla)p_0\right]$

By Divergence Thm $\underset{\uparrow}{=} \oint d\mathbf{S}\cdot[\boldsymbol{\xi}^*(\boldsymbol{\xi}\cdot\nabla)p_0] = \oint d\mathbf{S}\cdot[\boldsymbol{\xi}^*\underset{\uparrow}{\xi_x}p_0'] = 0$

NRL (28) p.5

By B.C.'s ξ_x at boundary ~~x~~ = 0, a is zero!
Periodic in y & z sums to zero.

6. Term ⑤: Only term left:

$$\boxed{\delta W = -\frac{1}{2}\int d^3\mathbf{x}\,|\xi_z|^2 g\rho_0'}$$

Lecture #3 (Continued)

II B. (Continued)

7. Thus, for $\rho_0' > 0$ (density increasing with height),

$$\delta W < 0 \Rightarrow \text{UNSTABLE!}$$