Lecture # 18 Review of Landau Damping; Cold Beam Instabilities Howes ①

I. Review: Landau Damping of Langmuir Waves

A. Electrostatic Approximation: Vlasov - Maxwell System

1. $\frac{\partial f_s}{\partial t} + \underset{\sim}{v} \cdot \nabla f_s - \frac{q_s}{m_s} \nabla \phi \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} = 0$

2. $-\nabla^2 \phi = \frac{1}{\epsilon_0} \sum_s \int d^3v \, q_s f_s$

B. Laplace Transform Approach by Landau

1. Linearization, Fourier Transform in Space, Laplace Transform in Time

$$\tilde{\phi}_1(\underset{\sim}{k}, p) = \frac{N(\underset{\sim}{k}, p)}{D(\underset{\sim}{k}, p)} \quad \text{where } N(\underset{\sim}{k}, p) = -i \sum_s \frac{q_s n_0}{\epsilon_0 k^3} \int_{-\infty}^{\infty} dv_z \frac{F_s(0)}{v_z - \frac{ip}{k}}$$

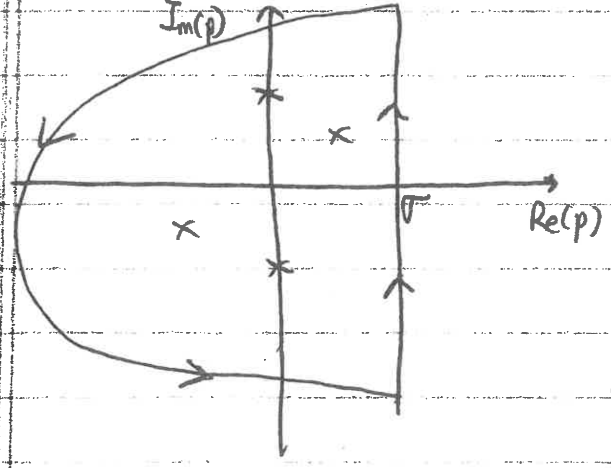
and the Dispersion Relation is $D(\underset{\sim}{k}, p) = 0$ where

$$\boxed{D(\underset{\sim}{k}, p) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\partial F_{s0}/\partial v_z}{v_z - ip/k}}$$

$D(\underset{\sim}{k}, p) = 0$ gives normal modes of system.

2. To solve, the Inverse Laplace Transform is

a. $$\phi_1(\underset{\sim}{k}, t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} dp \, \tilde{\phi}(\underset{\sim}{k}, p) e^{pt}$$

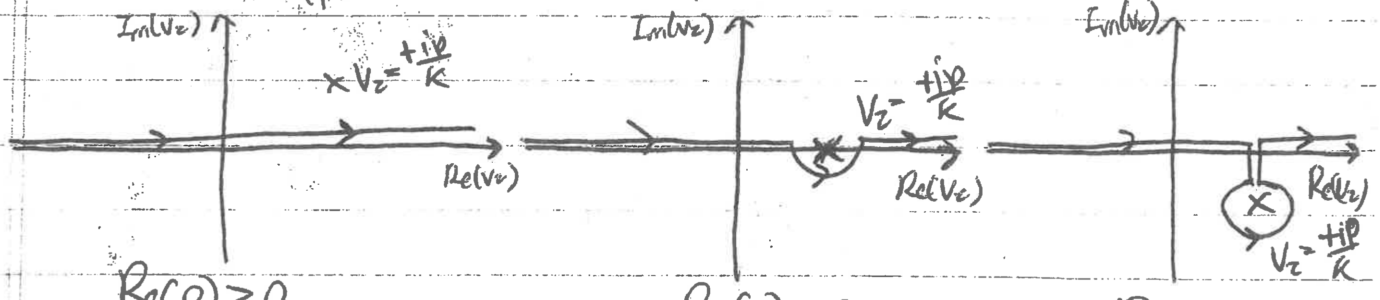


b. We evaluate this complex contour integral by closing the contour at $Re(p) \to -\infty$ and using the residue theorem.

c. To do so, we must analytically continue $\tilde{\phi}_1(\underset{\sim}{k}, p)$ from $Re(p) > 0$ to $Re(p) < 0$.

I. A. (Continued)

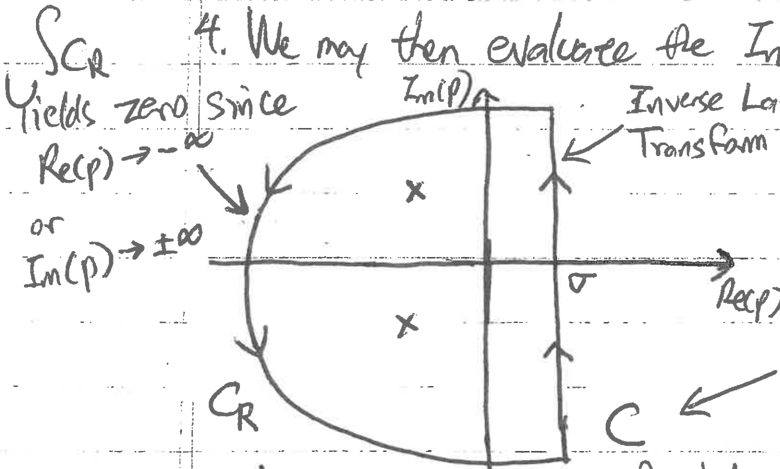
3. How do we analytically continue $D(\underline{k},p) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\partial F_{s0}/\partial v_z}{v_z - \frac{ip}{k}}$ from $Re(p) > 0$ to $Re(p) < 0$?



⇒ Contour __always__ passes below pole!

This definition yields an analytic form of $\tilde{\phi}(\underline{k},p)$ over entire complex p-plane (except for poles, of course).

4. We may then evaluate the Inverse Laplace Transform by Residue Theorem



a. $\phi_1(\underline{k},t) = \sum_j \underset{p=p_j}{\text{Res}} \left[\tilde{\phi}_1(\underline{k},p)e^{pt}\right]$

C. __Landau Damping of Waves__

1. For a __Cauchy Distribution__ of electrons with stationary ions,

a. $F_{0e}(v_z) = \frac{c}{\pi}\left(\frac{1}{c^2+v_z^2}\right)$

b. Dispersion Relation: $D(\underline{k},p) = 1 + \frac{\omega_p^2}{(p+|k|c)^2} = 0$

c. Wave Solutions with $\omega = \pm\omega_p$ and $\gamma = -|k|c$

Landau damping due to $Re(p) \neq 0$ (or $Im(\omega) \neq 0$)

I.C. (Continued)

2. Maxwellian Distribution:

a. The Plasma Dispersion Function

$$\boxed{Z(\xi_s) = \int_C \frac{dz}{\pi^{1/2}} \frac{e^{-\xi_s^2}}{z-\xi_s}}$$

where $\xi_s = \frac{ip}{k v_{ts}}$ or $\boxed{\xi_s = \frac{\omega}{k v_{ts}} + i\frac{\gamma}{k v_{ts}}}$

b. Using this function, the Langmuir Wave Dispersion Relation can be written,

$$\boxed{D(\underset{\sim}{k}, p) = 1 + \frac{1}{k^2 \lambda_{De}^2}\left[1 + \xi_e Z(\xi_e)\right] = 0}$$

c. In the high phase velocity limit, $|\xi_e| \gg 1$ or $\frac{\omega}{k} \gg v_{te}$, we find

$$\omega^2 = \omega_{pe}^2 + \frac{3}{2} k^2 v_{te}^2 \quad \text{and} \quad \gamma = -\sqrt{\frac{\pi}{8}} \frac{\omega_{pe}}{|k|^3 \lambda_{De}^3} e^{\left(\frac{-1}{2k^2\lambda_{De}^2} - \frac{3}{2}\right)}$$

Landau Damping

D. Physical Interpretation of Landau Damping

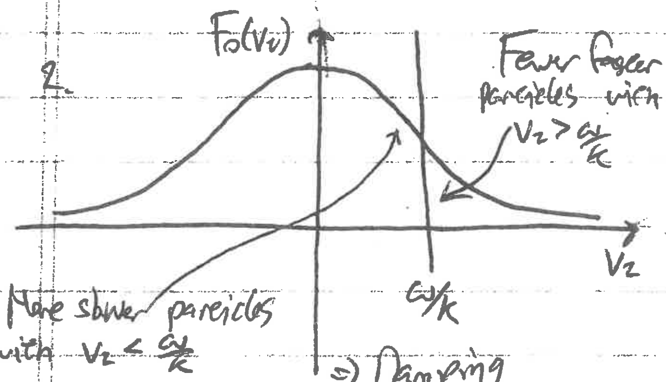
1. In the Weak Growth Rate Approximation, $|\gamma| \ll |\omega|$,

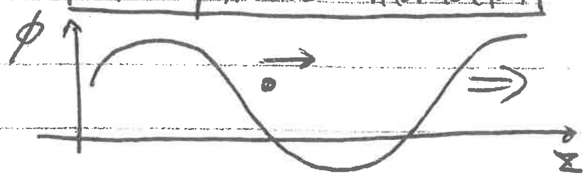
$$\gamma = \pi \frac{k}{|k|} \frac{1}{\partial D_r / \partial \omega} \sum_s \frac{\omega_{ps}^2}{k^2} \left.\frac{\partial F_{s0}}{\partial v_z}\right|_{v_z = \frac{\omega}{k}}$$

Damping is due to Resonant particles in distribution func.

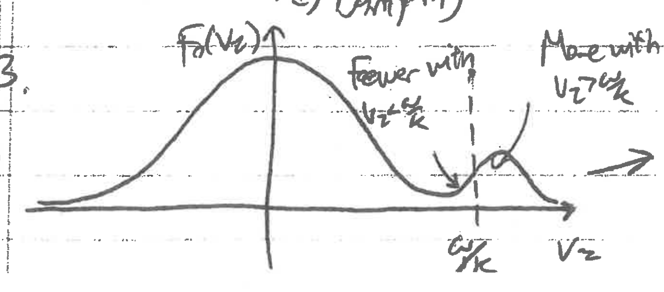
⇒ $\boxed{\text{Wave-particle Interaction}}$

2.





3.



Due to beam - fast particles. ⇒

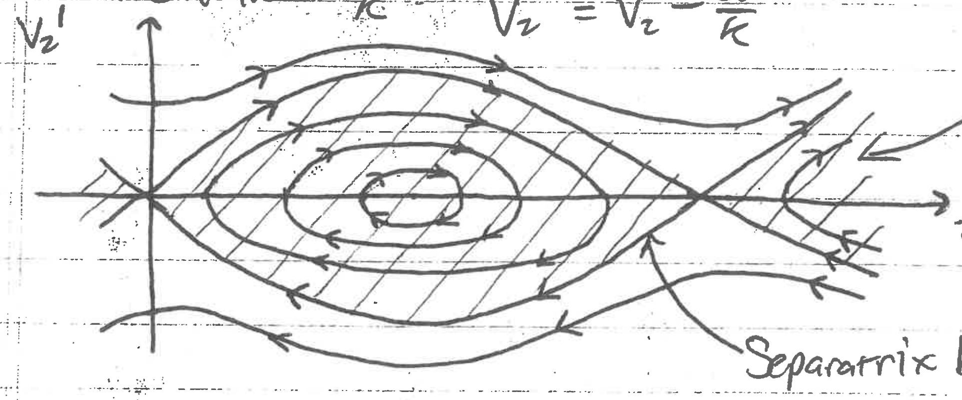
Bump-on-tail Instability:

Leads to unstable growth of $\underset{\sim}{E}$ fluctuations from free energy in $F_0(v_z)$.

II. Phase Mixing Interpretation

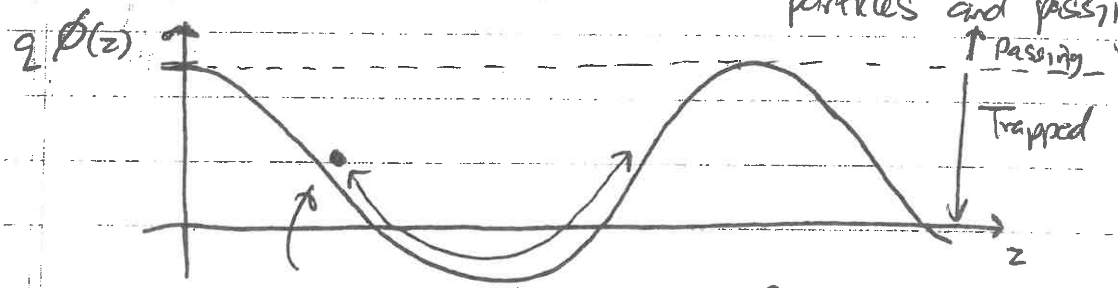
A. Phase Space Plot (v_z, z)

1. Transform to frame of reference moving at phase velocity of the wave $\frac{\omega}{k}$: $v_z' = v_z - \frac{\omega}{k}$



Trapped particles in shaded region of phase space.

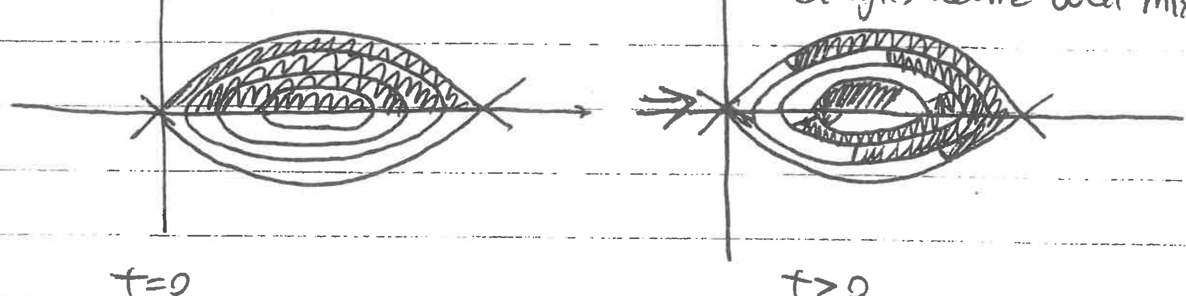
Separatrix between trapped particles and passing particles



Particles trapped in potential well (moving at phase speed $\frac{\omega}{k}$)

2. Bounce frequency of trapped particles decreases with increasing energy, reaching zero at separatrix.

3. Initially Particles are correlated — Later, particles at different energies become well mixed.



a. Since potential $-\nabla^2\phi = \sum_s \frac{1}{\epsilon_0}\int d^3v\, q_s f_s$ depends on integration over f_s, this phase mixing leads to an averaging out of the integral
$\Rightarrow$ damping of the wave.

III. Cold Beam Instabilities

A. Cold Beam Distribution:

1. A beam with number density n_j and velocity $\underline{v} = V_j \hat{\underline{z}}$ is given by

$$f_0(\underline{v}) = n_j \delta(v_x)\delta(v_y)\delta(v_z - V_j)$$

2. Since all particles have the same velocity, the beam has zero temperature $\Rightarrow$ no thermal spread of velocities.

3. For a species s, we can have multiple beams,

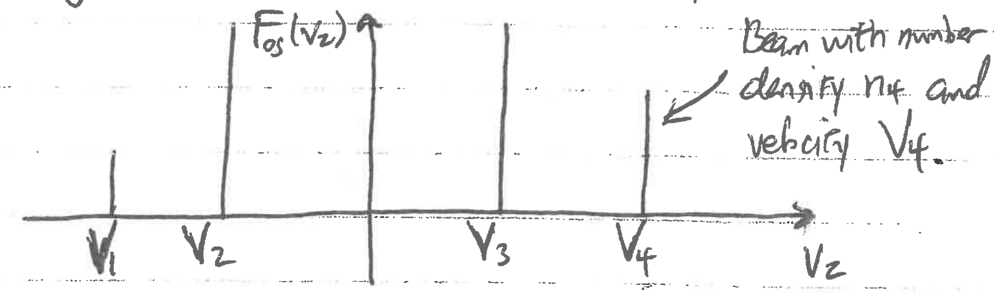
$$f_{0s}(\underline{v}) = \sum_j n_{js}\, \delta(v_x)\delta(v_y)\delta(v_z - V_{js})$$

4. The one-dimensional distribution function is written

$$F_{0s}(v_z) = \frac{1}{n_{0s}} \int_{-\infty}^{\infty} f_{0s}(\underline{v})\, dv_x\, dv_y = \sum_j \frac{n_{js}}{n_{0s}} \delta(v_z - V_j)$$

where $n_0 = \sum_j n_j$ is the total number density of all beams.

5. Distribution:



B. Dispersion Relation for Electrostatic Waves

1. The Laplace-Fourier Solution for electrostatic waves gives a Dispersion Relation

$$D(\underline{k}, p) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{F_{0s}}{(v_z - \frac{ip}{k})^2} = 0$$

2. Using $\sum_j \omega_{ps}^2 \frac{n_{js}}{n_{0s}} = \sum_j \frac{\cancel{n_{0s}} q_s^2}{\epsilon_0 m_s} \frac{n_{js}}{\cancel{n_{0s}}} = \sum_j \omega_{pj}^2$ where $\omega_{pj}^2 = \frac{n_{js} q_s^2}{\epsilon_0 m_s}$,

we obtain:

$$D(\underline{k}, p) = 1 - \sum_j \frac{\omega_{pj}^2}{k^2} \frac{1}{(V_{js} - \frac{ip}{k})^2} = 0$$

II.B. (Continued)

3. **NOTE**: Here we take a complex $\omega = ip$
(we may write $\omega = \omega_r + i\gamma$ to denote real & imaginary parts).

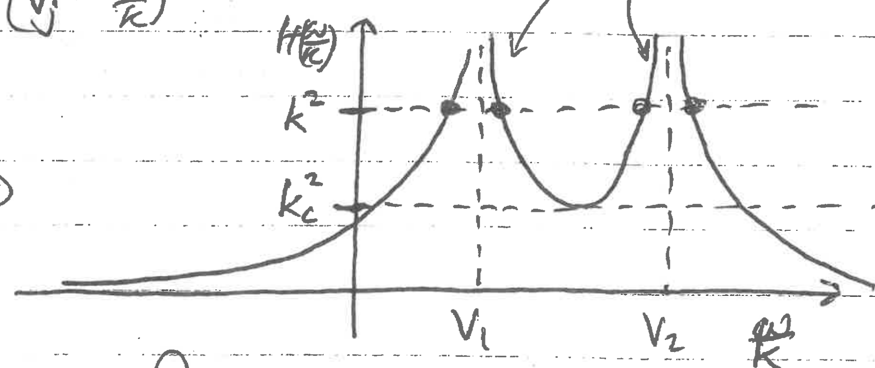
So
$$D(k,\omega) = 1 - \sum_j \frac{\omega_{pj}^2}{k^2} \frac{1}{\left(V_j - \frac{\omega}{k}\right)^2} = 0$$

4. Solutions occur when
$$H\left(\frac{\omega}{k}\right) = \sum_j \frac{\omega_{pj}^2}{\left(V_j - \frac{\omega}{k}\right)^2} = k^2$$

Two solutions per beam.

a. We may plot this as

For two beams $\Rightarrow$



b. For $k > k_c$, there exist four real solutions $\Rightarrow \gamma = 0$.

c. But, for $k < k_c$, one obtains two real & two imaginary solutions.
Since $H\left(\frac{\omega}{k}\right)$ is real, the roots come in complex conjugate pairs

$$\left.\begin{aligned} \omega_+ &= \omega_r + i\gamma \\ \omega_- &= \omega_r - i\gamma \end{aligned}\right\} \text{ Here we assume } \gamma > 0.$$

d. Thus, the root $\omega_+ = \omega_r + i\gamma$ gives a time dependence
$$\sim e^{-i\omega_r t} e^{+\gamma t}$$
Growth of this ω_+ solution. $\Rightarrow$ UNSTABLE

5. a. Single Beam leads to $\omega = \pm\omega_p - kV \Rightarrow$ Always stable.

b. When two (or more) beams are present, for sufficiently long wavelengths $k < k_c$, the plasma is always unstable.

$$\boxed{\text{Two Stream Instability}}$$

III. (Continued)

C. Two Stream Instability: Equal and opposite beams.

1. Take $n_1 = n_2 = n_0$ $\quad V_1 = -V_2 = V.$

2. Thus

$$D(k,\omega) = 1 - \frac{\omega_p^2}{(\omega - kV)^2} - \frac{\omega_p^2}{(\omega + kV)^2} = 0 \qquad \omega_p^2 = \frac{n_0 q_s^2}{\epsilon_0 m_s}$$

3. a. This equation is quadratic in ω^2 and can be expressed as

$$\omega^4 - \omega^2 2(\omega_p^2 + k^2V^2) - k^2V^2(2\omega_p^2 - k^2V^2) = 0$$

b. Solution: $$\boxed{\omega^2 = \underbrace{\omega_p^2 + k^2V^2}_{\text{positive definite}} \pm \underbrace{\sqrt{\omega_p^4 + 4\omega_p^2 k^2 V^2}}_{\text{Positive definite}}}$$

4. a. For solution with negative sign, when

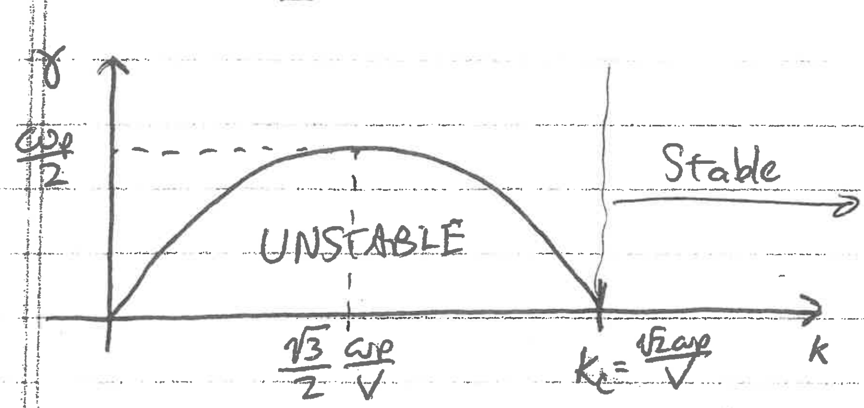
$$\omega_p^2 + k^2V^2 < (\omega_p^4 + 4\omega_p^2 k^2 V^2)^{\frac{1}{2}},$$

then $\omega^2 < 0$ and the solution $\omega = \pm i\gamma$, leading to unstable growth.

b. Condition for this is $\boxed{k < k_c = \sqrt{2}\,\frac{\omega_p}{V}}$

c. The growth rate is

$$\gamma = \left[(\omega_p^4 + 4k^2V^2\omega_p^2)^{\frac{1}{2}} - (\omega_p^2 + k^2V^2)\right]^{\frac{1}{2}}$$



III. D. Weak Beam Approximation

1. Take a low density electron beam in a plasma at rest, $n_b \ll n_0$.
2. For $\omega_b^2 = \frac{n_b q^2}{\epsilon_0 m}$, we get $D(\underset{\sim}{k},\omega) = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\epsilon \omega_b^2}{(kV-\omega)^2} = 0$
3. Ordering: $\epsilon = \frac{\omega_b^2}{\omega_p^2} \ll 1$: Let $\omega = \omega_0 + \epsilon\omega_1$.
   a. $\left[1 - \frac{\omega_p^2}{(\omega_0+\epsilon\omega_1)^2}\right](kV - (\omega_0+\epsilon\omega_1))^2 = \epsilon\omega_b^2$
   b. Expanding: $\frac{\omega_p^2}{\omega_0^2(1+\epsilon\frac{\omega_1}{\omega_0})^2} \simeq \frac{\omega_p^2}{\omega_0^2}\left(1 - \frac{2\epsilon\omega_1}{\omega_0}\right)$
4. $O(1)$: $\left(1 - \frac{\omega_p^2}{\omega_0^2}\right)(kV-\omega_0)^2 = 0 \Rightarrow \omega_0 = \pm\omega_p \quad k = \frac{\pm\omega_p}{V}$
5. To solve at $O(\epsilon)$, substitute solution ω_0 and k,
   a. $\left[1 - \cancel{\frac{\omega_p^2}{\omega_p^2}}^{1}\left(1 - \frac{2\epsilon\omega_1}{\omega_0}\right)\right]\left[\cancel{kV} - \cancel{\omega_p} - \omega_1\right]^2 = \omega_b^2$

   $\Rightarrow \boxed{\omega_1^3 = \frac{\omega_b^2\omega_p}{2}}$

   b. Three roots are: $\omega_1 = \left(\frac{1}{2}\right)^{\frac{1}{3}}\omega_b^{2/3}\omega_p^{\frac{1}{3}}$ and $\omega_1 = \left(\frac{1}{2}\right)^{\frac{1}{3}}\omega_b^{2/3}\omega_p^{\frac{1}{3}}\left(-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right)$
6. Root with positive sign gives unstable growth with
   a. $\omega_r = \omega_p\left[1 - \frac{1}{2}\left(\frac{1}{2}\right)^{\frac{1}{3}}\left(\frac{n_b}{n_0}\right)^{\frac{1}{3}}\right]$ and $\gamma = \frac{\sqrt{3}}{2}\left(\frac{1}{2}\right)^{\frac{1}{3}}\left(\frac{n_b}{n_0}\right)^{\frac{1}{3}}\omega_p$
   b. Growth rate $\gamma \propto n_b^{1/3}$

E. Buneman Instability

1. a. Background of cold ions at rest $\quad n_e = n_i = n_0$
   b. Cold "Beam" of electrons moving through ions.
2. Dispersion Relation: $D(\underset{\sim}{k},\omega) = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{(kV-\omega)^2} = 0$
3. Yields $\omega_r = \frac{1}{2}\left(\frac{m_e}{2m_i}\right)^{\frac{1}{3}}\omega_{pe} \qquad \gamma = \frac{\sqrt{3}}{2}\left(\frac{m_e}{2m_i}\right)^{\frac{1}{3}}\omega_{pe}$

This fast growing instability occurs when cold electrons stream through cold ions.