Lecture #3: Magnetic Diffusion and Intro to MHD Waves

Howes (1)

I. Magnetic Diffusion

A. Last time we studied the case of $R_{eM} \gg 1$, when resistivity can be neglected, giving

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B}).$$

From this equation, we proved the Frozen-in Flux Theorem:
The magnetic field lines are frozen to the fluid flow.

B. In the opposite limit, $R_{eM} \ll 1$, the convection term may be neglected, yielding a diffusion equation

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \frac{\eta}{\mu_0} \nabla^2 \underset{\sim}{B}$$

1. The timescale for the diffusion of the magnetic field τ_{diff} over a scale-length L can be estimated as

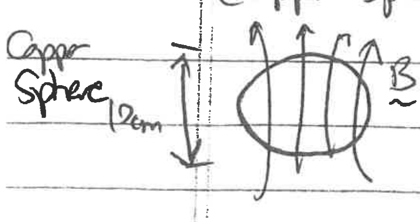
$$\frac{B}{\tau_{diff}} \sim \frac{\eta B}{\mu_0 L^2} \Rightarrow \boxed{\tau_{diff} \simeq \frac{\mu_0 L^2}{\eta}}$$

2. We can use this equation, along with the expression for resistivity

$$\eta = \frac{m_e \nu_{ei}}{e^2 n_0} = \frac{e^2 m_e^{\frac{1}{2}} \ln \Lambda_D}{2^{\frac{5}{2}} \pi \epsilon_0^2 (kT_e)^{\frac{3}{2}}} \quad \text{(from Lect \#11)}$$

to find the characteristic diffusion time in typical plasmas.

3. Given the resistivity of copper, $\eta = 1.7 \times 10^{-8}\ \Omega m$, a copper sphere of diameter 10 cm will diffuse a magnetic field



in $\tau_{diff} = \frac{\mu_0 L^2}{\eta} = \frac{(4\pi \times 10^{-7} H/m)(0.1 m)^2}{1.7 \times 10^{-8}\ \Omega \cdot m} = 0.7 s$

I. B. (Continued) NRL p. 40 has many characteristic values.

4.

Plasma	$n\,(m^{-3})$	$T\,(K)$	$B\,(T)$	$L\,(m)$	$\nu_{ei}\,(s^{-1})$	$\eta\,(\Omega\,m)$	τ_{diff}
LAPD	10^{18}	10^5	0.06	0.4	3×10^6	10^{-4}	$1.7\times10^{-3}\,s$
Fusion Plasma	10^{21}	10^8	10	2.0	2×10^5	6×10^{-9}	$8\times10^2\,s = 13\,m$
Solar Wind	10^7	10^5	10^{-8}	$1\,AU = 1.5\times10^{11}$	7×10^{-5}	2.5×10^{-4}	$5\times10^{19}\,s = 10^{12}\,y$
ISM	10^6	10^4	10^{-10}	$1\,pc = 3\times10^{16}\,m$	2×10^{-4}	7×10^{-3}	$2\times10^{29}\,s = 5\times10^{21}\,y$

a. NOTE: Although resistivities are larger than Copper (with the exception of the fusion plasma), diffusion times are ~~longer~~ long because of the scale of the plasma.

b. Space and astrophysical have very long characteristic diffusion times. Thus IDEAL MHD is a good approximation.

5. The earth's molten iron core has $\tau_{diff} \sim 10^4$ years.

a. Thus, earth's magnetic field must be maintained by some dynamo process!

6. Note also that the diffusion times depend only on plasma temperature T_e and density n. The magnitude of the magnetic field does not enter into the calculation.

II. Characteristic Waves of an MHD Plasma

A. Concept of Linear Wave Modes

1. A very important way of characterizing a plasma is to determine the characteristic linear wave modes, or eigenmodes, of the system.
2. A general perturbation (of small amplitude) can be decomposed into it's component linear waves modes. These waves will carry away the disturbance as the plasma response.

3. Linear Dispersion Relation
   a. IMPORTANT: the technique for determining the linear dispersion relation arises again and again in the study of plasma physics.
   b. The dispersion relation tells us a great deal about plasma behavior.

B. General Procedure for finding the Linear Dispersion Relation

1. Linearization of the Equations:
   a. We'll assume small amplitude perturbations so that quadratic terms will be negligible.
      Ex: Density: $\rho = \rho_0 + \epsilon \rho_1$ where $\epsilon \ll 1$.
      Magnetic Field $\underset{\sim}{B} = \underset{\sim}{B}_0 + \epsilon \underset{\sim}{B}_1$, etc.
   b. Plug these expansions into system of equations.
   c. Collect terms order by order
      i) Zeroth Order: $\mathcal{O}(\epsilon^0) = \mathcal{O}(1) \Rightarrow$ Plasma Equilibrium
      ii) First Order: $\mathcal{O}(\epsilon) \Rightarrow$ This gives the linearized equations.
      iii) Second Order: $\mathcal{O}(\epsilon^2) \Rightarrow$ Discard these non-linear terms.

Lecture # 3 (Continued)

II. B. (Continued)

2. Fourier Analysis:

a. Any disturbance can be decomposed into a sum of plane waves.

$$\rho(\underset{\sim}{x},t) = \sum_{\underset{\sim}{k}} \rho(\underset{\sim}{k})\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega(\underset{\sim}{k})t)}$$

Sum over all possible wave vectors $\underset{\sim}{k}$

This frequency is a function of $\underset{\sim}{k}$ to be determined by the dispersion relation.

b. Because the equations are now linear, each term has a sum, and each $\underset{\sim}{k}$ must solve that set of equations independent of ~~all~~ other wavevectors $\underset{\sim}{k}$!

c. Thus, linear properties of the system of equations (MHD) may be determined by the response to an arbitrary $\underset{\sim}{k}$. So, we take $\rho(\underset{\sim}{x},t) = \rho(\underset{\sim}{k})\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)}$ where $\omega = \omega(\underset{\sim}{k})$.

d. NOTE:

i) $\frac{\partial}{\partial t}\rho(\underset{\sim}{x},t) = \rho(\underset{\sim}{k})\frac{\partial}{\partial t} e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)} = -i\omega\, \rho(\underset{\sim}{k})\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)}$

$= -i\omega\, \rho(\underset{\sim}{x},t)$

ii) $\nabla \rho(\underset{\sim}{x},t) = \left(\frac{\partial}{\partial x}\hat{\underset{\sim}{x}} + \frac{\partial}{\partial y}\hat{\underset{\sim}{y}} + \frac{\partial}{\partial z}\hat{\underset{\sim}{z}}\right)\rho(\underset{\sim}{x},t)$

So $\hat{x}$ component: $\frac{\partial}{\partial x}\rho(\underset{\sim}{k})\, e^{i(k_x x + k_y y + k_z z - \omega t)} = i k_x \rho(\underset{\sim}{k})\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)}$

Thus

$\nabla \rho(\underset{\sim}{x},t) = i(k_x \hat{\underset{\sim}{x}} + k_y \hat{\underset{\sim}{y}} + k_z \hat{\underset{\sim}{z}})\rho(\underset{\sim}{k})\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)} = i\underset{\sim}{k}\, \rho(\underset{\sim}{k})\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)}$

iii) Therefore!

$$\boxed{\frac{\partial}{\partial t} \rightarrow -i\omega} \qquad \boxed{\nabla \rightarrow i\underset{\sim}{k}}$$

e. After substituting in for the plane wave (i.e. $\rho(\underset{\sim}{x},t) = \rho(\underset{\sim}{k})\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)}$), we can cancel $e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)}$ from each term to give a system of equations for $\rho(\underset{\sim}{k})$, $\underset{\sim}{B}(\underset{\sim}{k})$, etc.

Lecture #3 (Continued) Howes (5)

II. B.2 (Continued)

f. Complex Notation: i) The coefficient $\rho(\underset{\sim}{k})$ is taken to be complex.

ii) The observable quantity is $Re[\rho(\underset{\sim}{k}) e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x}-\omega t)}]$.

iii) If $\rho(\underset{\sim}{k})$ were real, this would be

$$\rho(\underset{\sim}{k}) \cos(\underset{\sim}{k}\cdot\underset{\sim}{x}-\omega t)$$

iv) But, since $\rho(\underset{\sim}{k})$ is complex, the real part allows for arbitrary phase,

$$\rho_r(\underset{\sim}{k}) \cos(\underset{\sim}{k}\cdot\underset{\sim}{x}-\omega t) - \rho_i(\underset{\sim}{k}) \sin(\underset{\sim}{k}\cdot\underset{\sim}{x}-\omega t)$$

v) This is equivalent to allowing an arbitrary phase δ, such that

$$\underbrace{\rho(\underset{\sim}{k})}_{\text{Real Constant}} e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x}-\omega t+\delta)} = \underbrace{\rho(\underset{\sim}{k}) e^{i\delta}}_{\text{Complex Constant}} e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x}-\omega t)}$$

3. Collect System of linear equations for Fourier Amplitudes

a. Assemble system of linear equations into Matrix Form.

$$\underbrace{\left(\quad\quad\quad\quad\right)}_{N\times N \text{ matrix}} \left(\quad\right) = 0$$

Vector of N-variables (N=8 for MHD)

b. $\boxed{\text{Determinant of } N\times N \text{ matrix} = 0}$

This yields solubility condition for system of equations

c. This yields the Dispersion Relation of the form

$$\omega = \omega(\underset{\sim}{k})$$

d. There may be other physical system parameters on which ω depends.

Lecture # 3 (Continued)

II. (Continued)

C. General Properties of MHD Dispersion Relation

1. Basic properties of plane wave solutions

a. Consider a wavevector $\mathbf{k} = k_\parallel \hat{\mathbf{z}}$ and dispersion relation $\omega = k_\parallel V_A$.

i) $e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} = e^{i(k_\parallel z - k_\parallel V_A t)} = e^{i k_\parallel (z - V_A t)}$

ii) This wave has constant phase at $z - V_A t = \text{const}$ or $z = V_A t + \text{const}$. The wave is moving in $+\hat{\mathbf{z}}$ direction at speed V_A.

iii) If $\omega = -k_\parallel V_A$, then wave moves in $-\hat{\mathbf{z}}$ direction with speed V_A.

b. Phase velocity: DEF: $\boxed{\mathbf{V}_p = \frac{\omega}{\mathbf{k}}} = \frac{\omega}{k_x}\hat{\mathbf{x}} + \frac{\omega}{k_y}\hat{\mathbf{y}} + \frac{\omega}{k_z}\hat{\mathbf{z}}$

Ex: For $\mathbf{k} = k_\parallel \hat{\mathbf{z}}$ and $\omega = k_\parallel V_A$, $\mathbf{V}_p = \frac{\omega}{\mathbf{k}} = \frac{k_\parallel V_A}{k_\parallel}\hat{\mathbf{z}} = V_A \hat{\mathbf{z}}$

c. Group velocity: This is the velocity at which information (and energy) propagates.

DEF: $\boxed{\mathbf{V}_g = \frac{d\omega}{d\mathbf{k}}} = \frac{d\omega}{dk_x}\hat{\mathbf{x}} + \frac{d\omega}{dk_y}\hat{\mathbf{y}} + \frac{d\omega}{dk_z}\hat{\mathbf{z}}$

Ex: For same example above,

$$\mathbf{V}_g = \frac{d\omega}{d\mathbf{k}} = \frac{d}{dk_\parallel}(k_\parallel V_A)\hat{\mathbf{z}} = V_A \hat{\mathbf{z}}$$

2. Axisymmetry of MHD Equations.

a. In a plasma with a straight, uniform magnetic field $\mathbf{B}_0 = B_0 \hat{\mathbf{b}}$, there are three distinct axes for a wave mode with wavevector $\mathbf{k}$:

$$\boxed{\hat{\mathbf{k}}_\perp, \hat{\mathbf{b}}, \hat{\mathbf{k}}_\perp \times \hat{\mathbf{b}}}$$

where $\mathbf{k} = k_\parallel \hat{\mathbf{b}} + \mathbf{k}_\perp$

b. The angle of $\mathbf{k}_\perp$ w.r.t. $\hat{\mathbf{b}}$ is arbitrary, so there is an axis of symmetry.

III. The MHD Dispersion Relation

A. Begin with the Ideal MHD System of Equations

Continuity $\frac{\partial \rho}{\partial t} + \underset{\sim}{U} \cdot \nabla \rho = -\rho \nabla \cdot \underset{\sim}{U}$

Momentum $\rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho \underset{\sim}{U} \cdot \nabla \underset{\sim}{U} = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{(\underset{\sim}{B}\cdot\nabla)\underset{\sim}{B}}{\mu_0}$

Induction $\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B})$

Pressure $\frac{\partial p}{\partial t} + \underset{\sim}{U} \cdot \nabla p = -\gamma p \nabla \cdot \underset{\sim}{U}$

B. Linearize Equations: Take Uniform $\underset{\sim}{B}$ field in homogeneous plasma with no mean flow.

1. Take a.
$$\left.\begin{aligned} \rho &= \rho_0 + \epsilon \rho_1 \\ \underset{\sim}{B} &= \underset{\sim}{B}_0 + \epsilon \underset{\sim}{B}_1 \\ \underset{\sim}{U} &= \epsilon \underset{\sim}{U}_1 \\ p &= p_0 + \epsilon p_1 \end{aligned}\right\} \text{ where } \epsilon \ll 1$$

b. Let ρ_0, $\underset{\sim}{B}_0$, and p_0 be uniform in space and constant in time.

2. Substitute into equations:

a. $\cancel{\frac{\partial \rho_0}{\partial t}}^{0} + \epsilon \frac{\partial \rho_1}{\partial t} + \epsilon \cancel{\underset{\sim}{U}_1 \cdot \nabla \rho_0}^{0} + \epsilon^2 \underset{\sim}{U}_1 \cdot \nabla \rho_1 = -\epsilon \rho_0 \nabla \cdot \underset{\sim}{U}_1 - \epsilon^2 \rho_1 \nabla \cdot \underset{\sim}{U}_1$

$O(\epsilon)$: $\boxed{\frac{\partial \rho_1}{\partial t} = -\rho_0 \nabla \cdot \underset{\sim}{U}_1}$

b. $\epsilon \rho_0 \frac{\partial \underset{\sim}{U}_1}{\partial t} + \epsilon^2 \rho_1 \frac{\partial \underset{\sim}{U}_1}{\partial t} + \epsilon^2 \underset{\sim}{U}_1 \cdot \nabla \underset{\sim}{U}_1 = -\cancel{\nabla p_0}^{0} - \epsilon \nabla p_1 - \cancel{\nabla \frac{|B_0|^2}{2\mu_0}}^{0} - \epsilon \nabla \frac{\underset{\sim}{B}_0 \cdot \underset{\sim}{B}_1}{\mu_0} - \epsilon^2 \nabla \frac{|B_1|^2}{2\mu}$
$+ \cancel{\frac{\underset{\sim}{B}_0 \cdot \nabla \underset{\sim}{B}_0}{\mu_0}}^{0} + \epsilon \cancel{\frac{\underset{\sim}{B}_1 \cdot \nabla \underset{\sim}{B}_0}{\mu_0}}^{0} + \epsilon \frac{\underset{\sim}{B}_0 \cdot \nabla \underset{\sim}{B}_1}{\mu_0} + \epsilon^2 \frac{\underset{\sim}{B}_1 \cdot \nabla \underset{\sim}{B}_1}{\mu_0}$

$O(\epsilon)$: $\boxed{\rho_0 \frac{\partial \underset{\sim}{U}_1}{\partial t} = -\nabla\left(p_1 + \frac{\underset{\sim}{B}_0 \cdot \underset{\sim}{B}_1}{\mu_0}\right) + \frac{\underset{\sim}{B}_0 \cdot \nabla \underset{\sim}{B}_1}{\mu_0}}$

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II B (Continued)

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c. $$\cancel{\frac{\partial \mathbf{B}_0}{\partial t}}^{\,0} + \epsilon \frac{\partial \mathbf{B}_1}{\partial t} = \epsilon \nabla\times(\mathbf{U}_1\times \mathbf{B}_0) + \cancel{\epsilon^2 \nabla\times(\mathbf{U}_1\times \mathbf{B}_1)}$$

$$O(\epsilon): \quad \frac{\partial \mathbf{B}_1}{\partial t} = \nabla\times(\mathbf{U}_1\times\mathbf{B}_0) = \cancel{\mathbf{U}_1 \nabla\cdot\mathbf{B}_0}^{\,0} - \mathbf{B}_0 \nabla\cdot\mathbf{U}_1 + \mathbf{B}_0\cdot\nabla\mathbf{U}_1 - \cancel{\mathbf{U}_1\cdot\nabla\mathbf{B}_0}^{\,0}$$

NRL p.4 (10)

$$\boxed{\frac{\partial \mathbf{B}_1}{\partial t} = -\mathbf{B}_0 \nabla\cdot\mathbf{U}_1 + \mathbf{B}_0\cdot\nabla\mathbf{U}_1}$$

d. $$\cancel{\frac{\partial p_0}{\partial t}}^{\,0} + \epsilon\frac{\partial p_1}{\partial t} + \cancel{\epsilon \mathbf{U}_1\cdot\nabla p_0}^{\,0} + \epsilon^2 \mathbf{U}_1\cdot\nabla p_1 = -\epsilon\gamma p_0 \nabla\cdot\mathbf{U}_1 - \epsilon^2\gamma p_1 \nabla\cdot\mathbf{U}_1$$

$$O(\epsilon): \quad \boxed{\frac{\partial p_1}{\partial t} = -\gamma p_0 \nabla\cdot\mathbf{U}_1}$$

C. Fourier Analysis: Take plane wave solutions $\sim e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$

1. $-i\omega\rho_1 = -\rho_0\, i(\mathbf{k}\cdot\mathbf{U}_1) \Rightarrow \boxed{\omega\rho_1 = \rho_0(\mathbf{k}\cdot\mathbf{U}_1)}$

2. $-i\omega\rho_0\mathbf{U}_1 = -i\mathbf{k}\left(p_1 + \frac{\mathbf{B}_0\cdot\mathbf{B}_1}{\mu_0}\right) + \frac{i(\mathbf{B}_0\cdot\mathbf{k})\mathbf{B}_1}{\mu_0} \Rightarrow \boxed{\omega\mathbf{U}_1 = \mathbf{k}\left(\frac{p_1}{\rho_0} + \frac{\mathbf{B}_0\cdot\mathbf{B}_1}{\mu_0\rho_0}\right) - \frac{(\mathbf{B}_0\cdot\mathbf{k})\mathbf{B}_1}{\mu_0\rho_0}}$

3. $-i\omega\mathbf{B}_1 = -i\mathbf{B}_0(\mathbf{k}\cdot\mathbf{U}_1) + i(\mathbf{B}_0\cdot\mathbf{k})\mathbf{U}_1 \Rightarrow \boxed{\omega\mathbf{B}_1 = \mathbf{B}_0(\mathbf{k}\cdot\mathbf{U}_1) - (\mathbf{B}_0\cdot\mathbf{k})\mathbf{U}_1}$

4. $-i\omega p_1 = -i\gamma p_0(\mathbf{k}\cdot\mathbf{U}_1) \Rightarrow \boxed{\omega p_1 = \gamma p_0(\mathbf{k}\cdot\mathbf{U}_1)}$

5. Thus, we have found:

$$\boxed{\begin{aligned}
\omega\rho_1 &= \rho_0(\mathbf{k}\cdot\mathbf{U}_1)\\
\omega\mathbf{U}_1 &= \mathbf{k}\left(\frac{p_1}{\rho_0} + \frac{\mathbf{B}_0\cdot\mathbf{B}_1}{\mu_0\rho_0}\right) - \frac{(\mathbf{B}_0\cdot\mathbf{k})\mathbf{B}_1}{\mu_0\rho_0}\\
\omega\mathbf{B}_1 &= \mathbf{B}_0(\mathbf{k}\cdot\mathbf{U}_1) - (\mathbf{B}_0\cdot\mathbf{k})\mathbf{U}_1\\
\omega p_1 &= \gamma p_0(\mathbf{k}\cdot\mathbf{U}_1)
\end{aligned}}$$

Next time we'll finish solving for the linear MHD dispersion relation