Lecture #4 MHD Waves: Alfven, Fast, & Slow Waves

I. Review

A. Calculation of Linear Dispersion Relation

1. Linearize System of Equations (e.g., assume $\rho = \rho_0 + \epsilon \rho_1$ where $\epsilon \ll 1$)
2. Fourier Analysis: Find plane wave solutions $\sim e^{i(\underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t)}$
3. Write system of equations as matrix equation
   $\Rightarrow$ Solve Determinant $= 0$ to yield $\omega = \omega(\underset{\sim}{k})$.

B. We left off with the following system

① $\omega \rho_1 = \rho_0 (\underset{\sim}{k} \cdot \underset{\sim}{U}_1)$

② $\omega \underset{\sim}{U}_1 = \underset{\sim}{k}\left(\frac{p_1}{\rho_0} + \frac{\underset{\sim}{B}_0 \cdot \underset{\sim}{B}_1}{\mu_0 \rho_0}\right) - \frac{(\underset{\sim}{B}_0 \cdot \underset{\sim}{k}) \underset{\sim}{B}_1}{\mu_0 \rho_0}$ (Note missing term in previous notes)

③ $\omega \underset{\sim}{B}_1 = \underset{\sim}{B}_0 (\underset{\sim}{k} \cdot \underset{\sim}{U}_1) - (\underset{\sim}{B}_0 \cdot \underset{\sim}{k}) \underset{\sim}{U}_1$

④ $\omega p_1 = \gamma p_0 (\underset{\sim}{k} \cdot \underset{\sim}{U}_1)$

II. The MHD Dispersion Relation (Continued)

A. Let's simplify these equations

1. Eliminate p_1 from ② using ④

$$② \Rightarrow ⓐ \quad \omega^2 \underset{\sim}{U}_1 = \underset{\sim}{k}\left(\frac{[\gamma p_0 (\underset{\sim}{k} \cdot \underset{\sim}{U}_1)]}{\rho_0} + \frac{\underset{\sim}{B}_0 \cdot [\omega \underset{\sim}{B}_1]}{\mu_0 \rho_0}\right) - \frac{(\underset{\sim}{B}_0 \cdot \underset{\sim}{k})[\omega \underset{\sim}{B}_1]}{\mu_0 \rho_0}$$

2. Now, use ③ to substitute for $\omega \underset{\sim}{B}_1$ in 2a

$$2a \Rightarrow 2b \quad \omega^2 \underset{\sim}{U}_1 = \underset{\sim}{k} \frac{\gamma p_0}{\rho_0}(\underset{\sim}{k} \cdot \underset{\sim}{U}_1) + \underset{\sim}{k} \frac{\underset{\sim}{B}_0 \cdot \underset{\sim}{B}_0 (\underset{\sim}{k} \cdot \underset{\sim}{U}_1)}{\mu_0 \rho_0} - \underset{\sim}{k} \frac{(\underset{\sim}{B}_0 \cdot \underset{\sim}{U}_1)(\underset{\sim}{B}_0 \cdot \underset{\sim}{k})}{\mu_0 \rho_0}$$

$$- \frac{(\underset{\sim}{B}_0 \cdot \underset{\sim}{k})(\underset{\sim}{k} \cdot \underset{\sim}{U}_1) \underset{\sim}{B}_0}{\mu_0 \rho_0} + \frac{(\underset{\sim}{B}_0 \cdot \underset{\sim}{k})^2 \underset{\sim}{U}_1}{\mu_0 \rho_0}$$

3. Use $\underset{\sim}{B}_0 = B_0 \hat{\underset{\sim}{b}}$ to simplify further:

$$\omega^2 \underset{\sim}{U}_1 = \underset{\sim}{k}(\underset{\sim}{k} \cdot \underset{\sim}{U}_1)\left[\frac{\gamma p_0}{\rho_0} + \frac{B_0^2}{\mu_0 \rho_0}\right] - \underset{\sim}{k} \frac{(\hat{\underset{\sim}{b}} \cdot \underset{\sim}{U}_1)(\hat{\underset{\sim}{b}} \cdot \underset{\sim}{k}) B_0^2}{\mu_0 \rho_0} - \frac{B_0^2}{\mu_0 \rho_0}(\hat{\underset{\sim}{b}} \cdot \underset{\sim}{k})(\underset{\sim}{k} \cdot \underset{\sim}{U}_1)\hat{\underset{\sim}{b}}$$

$$+ \frac{B_0^2 (\hat{\underset{\sim}{b}} \cdot \underset{\sim}{k})^2 \underset{\sim}{U}_1}{\mu_0 \rho_0}$$

Lecture #4 (Continued) Waves ②

II. A. (Continued)

4. Define: DEF: Sound Speed $c_s^2 = \frac{\gamma p_0}{\rho_0}$

Alfven Speed $V_A^2 = \frac{B_0^2}{\mu_0 \rho_0}$

5.

$$\boxed{\omega^2 \underset{\sim}{U_1} = (c_s^2 + V_A^2)\underset{\sim}{k}(\underset{\sim}{k}\cdot\underset{\sim}{U_1}) - V_A^2(\hat{\underset{\sim}{b}}\cdot\underset{\sim}{U_1})(\hat{\underset{\sim}{b}}\cdot\underset{\sim}{k})\underset{\sim}{k} - V_A^2(\hat{\underset{\sim}{b}}\cdot\underset{\sim}{k})(\underset{\sim}{k}\cdot\underset{\sim}{U_1})\hat{\underset{\sim}{b}} + V_A^2(\hat{\underset{\sim}{b}}\cdot\underset{\sim}{k})^2\underset{\sim}{U_1}}$$

a. Thus, we have reached a single (vector) equation for $\underset{\sim}{U_1}$.

b. NOTE: Once we have solved for $\underset{\sim}{U_1}$, ρ_1 is determined by $\underset{\sim}{U_1}$ using equation ①.

c. This vector equation represent 3 component equations. Thus, we can simplify to a matrix form:

$$\begin{pmatrix} & & \\ & 3\times3 \text{ matrix.} & \\ & & \end{pmatrix}\begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix} = 0$$

But, first, we'll explore the solutions in simplified limits.

B. $\boxed{\text{MHD Waves for } \underset{\sim}{k} = k_{\parallel}\hat{\underset{\sim}{b}}}$ (Parallel wave vector $\underset{\sim}{k} \parallel \underset{\sim}{B_0}$)

1. In this case, we can simplify: $(\underset{\sim}{k}\cdot\underset{\sim}{U_1}) = k_{\parallel} U_z$

$$\hat{\underset{\sim}{b}}\cdot\underset{\sim}{U_1} = U_z$$

$$\hat{\underset{\sim}{b}}\cdot\underset{\sim}{k} = k_{\parallel}$$

where we take $\hat{\underset{\sim}{b}} = \hat{\underset{\sim}{z}}$ and $\underset{\sim}{U_1} = U_x\hat{x} + U_y\hat{y} + U_z\hat{\underset{\sim}{z}}$.

2. Thus,

$$\omega^2 \underset{\sim}{U_1} = (c_s^2 + \cancel{V_A^2})k_{\parallel}^2 U_z \hat{\underset{\sim}{b}} \cancel{- V_A^2 k_{\parallel}^2 U_z \hat{\underset{\sim}{b}}} - V_A^2 k_{\parallel}^2 U_z \hat{\underset{\sim}{b}} + k_{\parallel}^2 V_A^2 \underset{\sim}{U_1}$$

Lecture #4 (Continued)

II. B. (Continued)

3. Splitting into components and putting into matrix form

$$\begin{pmatrix} \omega^2 - k_\parallel^2 V_A^2 & 0 & 0 \\ 0 & \omega^2 - k_\parallel^2 V_A^2 & 0 \\ 0 & 0 & \omega^2 - k_\parallel^2 c_s^2 \end{pmatrix} \begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix} = 0$$

4. The determinant $D=0$ is dispersion relation:

$$\boxed{(\omega^2 - k_\parallel^2 V_A^2)^2 (\omega^2 - k_\parallel^2 c_s^2) = 0}$$

5. There are six solutions to this equation:

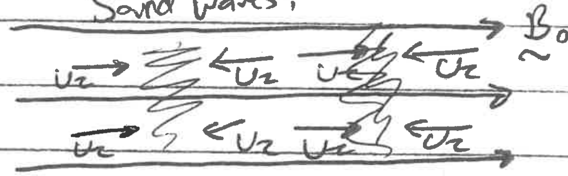
a. $\omega = \pm k_\parallel V_A$] ×2, one for $U_x \neq 0, U_y = 0$, another for $U_x = 0, U_y \neq 0$.

b. $\omega = \pm k_\parallel c_s$

6. Parallel Motions: $\boxed{\text{Sound Waves}}$

a. If we have $U_z \neq 0$, then $\boxed{\omega = \pm k_\parallel c_s}$

Sound waves!



b. These are the usual sound waves motion along B_0 at sound speed $c_s = \sqrt{\frac{\gamma p_0}{\rho_0}}$

c. Since $U_z = U_{z0} e^{i(\underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t)} = U_{z0} e^{i k_\parallel (z \pm c_s t)}$

d. $\underset{\sim}{B}$ is unperturbed by motion along B_0.

e. In this limit of $\underset{\sim}{k} = k_\parallel \hat{\underset{\sim}{b}}$, the relevant equations are ~~[illegible]~~

$\hat{z}$-component of Momentum eq: $\rho_0 \frac{\partial U_z}{\partial t} = -\frac{\partial}{\partial z} p_1$

Pressure equation: $\frac{\partial p_1}{\partial t} = -\gamma p_0 \frac{\partial U_z}{\partial z}$

Parallel motion U_z leads to compression Pressure p_1 acts as restoring force	} the usual sound wave!

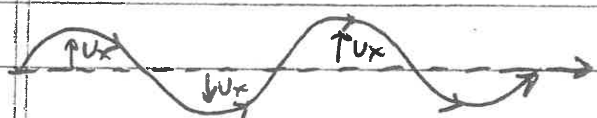
Lecture #4 (Continued)

Howes ④

II. B. (Continued)

$\boxed{\text{Alfven Waves}}$

7. Perpendicular Motions: a. For $U_x \neq 0$, we must have $\boxed{\omega = \pm k_\parallel V_A}$



Alfven speed.

b. Alfven waves are like waves on a string, propagating at $\boxed{V_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}}}$

c. Relevant equations:

$\hat{x}$-component of Momentum Eq: $\rho_0 \frac{\partial U_x}{\partial t} = \frac{B_0}{\mu_0} \frac{\partial B_x}{\partial z}$ ← Magnetic Tension ~~term~~.

$\hat{x}$-component of Induction Eq: $\frac{\partial B_x}{\partial t} = B_0 \frac{\partial U_x}{\partial z}$

Motion U_x is perpendicular to $\underset{\sim}{B_0}$, causing it to bend Magnetic Tension acts as restoring force

d. Because $\underset{\sim}{k} \cdot \underset{\sim}{U_1} = 0$, this motion is incompressible.

e. We could also have taken $U_y \neq 0$ with $U_x = 0$, and results are analogous. ~~There~~

Two polarizations of Alfven Wave in direction perpendicular to $\underset{\sim}{B_0}$

C. $\boxed{\text{MHD Waves for } \underset{\sim}{k} = \underset{\sim}{k_\perp} = k_\perp \hat{x}}$ (Perpendicular Wavevector $\underset{\sim}{k_\perp} \perp \underset{\sim}{B_0}$)

1. In this case
$$(\underset{\sim}{k} \cdot \underset{\sim}{U_1}) = k_\perp U_x$$
$$(\hat{\underset{\sim}{b}} \cdot \underset{\sim}{U_1}) = U_z$$
$$(\hat{\underset{\sim}{b}} \cdot \underset{\sim}{k}) = 0$$

2. Thus $\omega^2 \underset{\sim}{U_1} = (c_s^2 + v_A^2) k_\perp^2 U_x \hat{x} + 0$

Lecture #4 (Continued) Hawes (5)

II. C. (Continued)

3. Splitting into Component Form:

$$\begin{pmatrix} \omega^2 - k_\perp^2(c_s^2+V_A^2) & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}\begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix} = 0$$

4. The determinant $D=0$ gives

$$\omega^4\left[\omega^2 - k_\perp^2(c_s^2+V_A^2)\right]=0$$

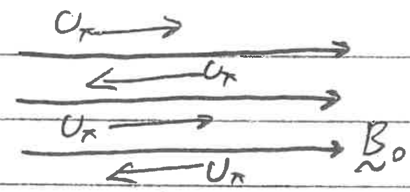
5. Again, we have six solutions:

a. Four solutions with $\omega = 0$ for $U_y \neq 0$ or $U_z \neq 0$.

b. Two solutions with $\omega = \pm k_\perp (c_s^2+V_A^2)^{\frac{1}{2}}$ $U_x \neq 0$

6. <u>Zero Frequency Solutions:</u>

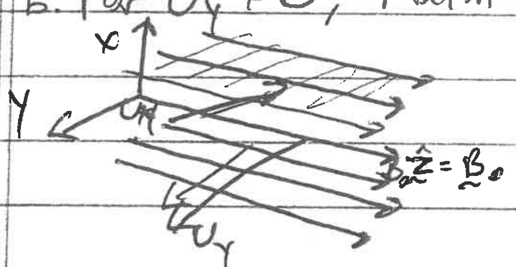
a. For $U_z \neq 0$,



i. These are like sound waves (motion along field)

ii. But, $\underset{\sim}{k} \cdot \underset{\sim}{U} = 0$, so no compression, and thus no restoring force.

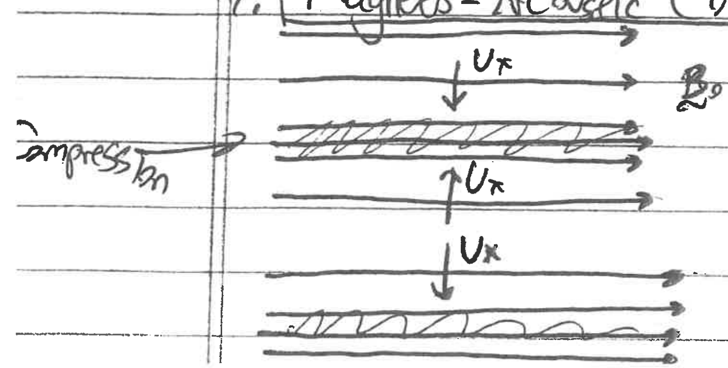
b. For $U_y \neq 0$, Motion is in $\hat{y} = \underset{\sim}{B}_0 \times \underset{\sim}{k}_\perp$ direction.



i) Magnetic field lines may slide past one another. Again, no restoring force, so $\omega = 0$.

ii) This is called an <u>interchange motion</u>: It moves straight magnetic field lines with bending them.

7. $\boxed{\text{Magneto-Acoustic (or Fast) Wave}}$ $U_x \neq 0 \Rightarrow \omega = \pm k_\perp (c_s^2+V_A^2)^{\frac{1}{2}}$



a. Motions are similar to compressional sound waves, but include a contribution from the magnetic pressure as well, propagating at speed $(c_s^2+V_A^2)^{\frac{1}{2}}$.

Lecture #4 (Continued)

II.C (Continued)

b. Relevant Equations:

$\hat{x}$- Component of Momentum Eq.
$$\rho_0 \frac{\partial U_x}{\partial t} = -\frac{\partial}{\partial x}\Big(\overbrace{p_1}^{\text{Thermal Pressure}} + \overbrace{\frac{B_0}{\mu_0} B_z}^{\text{Magnetic Pressure}}\Big)$$

$\hat{z}$- Component of Induction Eq.
$$\frac{\partial B_z}{\partial t} = -B_0 \frac{\partial U_x}{\partial x}$$

Pressure Equation
$$\frac{\partial p_1}{\partial t} = -\gamma p_0 \frac{\partial U_x}{\partial x}$$

c. $\underset{\sim}{k} \cdot \underset{\sim}{U}_1 = k_\perp U_{1x} \neq 0$, so these waves are compressional.

d. Perpendicular motion U_x compresses both plasma and magnetic field.
Restoring force includes both thermal pressure p_1
and magnetic pressure due to B_z.

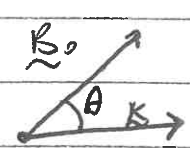
e. NOTE: A fluctuation with only $B_z \neq 0$ has $\underset{\sim}{B} = (B_0 + B_z)\hat{\underset{\sim}{z}}$.
Magnetic field does not change direction
but does increase magnitude.

D. The General Case of MHD Dispersion Relation

1. We can solve the MHD Dispersion Relation for any wavevector $\underset{\sim}{k}$.

a. Without loss of generality, we take
$$\underset{\sim}{k} = k_\perp \hat{\underset{\sim}{x}} + k_\parallel \hat{\underset{\sim}{z}} = k\sin\theta\, \hat{\underset{\sim}{x}} + k\cos\theta\, \hat{\underset{\sim}{z}}$$

where $\underset{\sim}{k} \cdot \hat{\underset{\sim}{b}} = k\cos\theta$



b. In general, then,
$$\underset{\sim}{k} \cdot \underset{\sim}{U}_1 = k\sin\theta\, U_x + k\cos\theta\, U_z$$
$$\hat{\underset{\sim}{b}} \cdot \underset{\sim}{k} = k\cos\theta$$
$$\underset{\sim}{U}_1 \cdot \hat{\underset{\sim}{b}} = U_z$$

Lecture #4

II. D (Continued)

2. After some algebra, the dispersion relation is found to be:

$$\boxed{\left(\omega^2 - k^2\cos^2\theta\, V_A^2\right)\left[\omega^4 - \omega^2 k^2 (C_s^2 + V_A^2) + k^4 \cos^2\theta\, C_s^2 V_A^2\right] = 0}$$

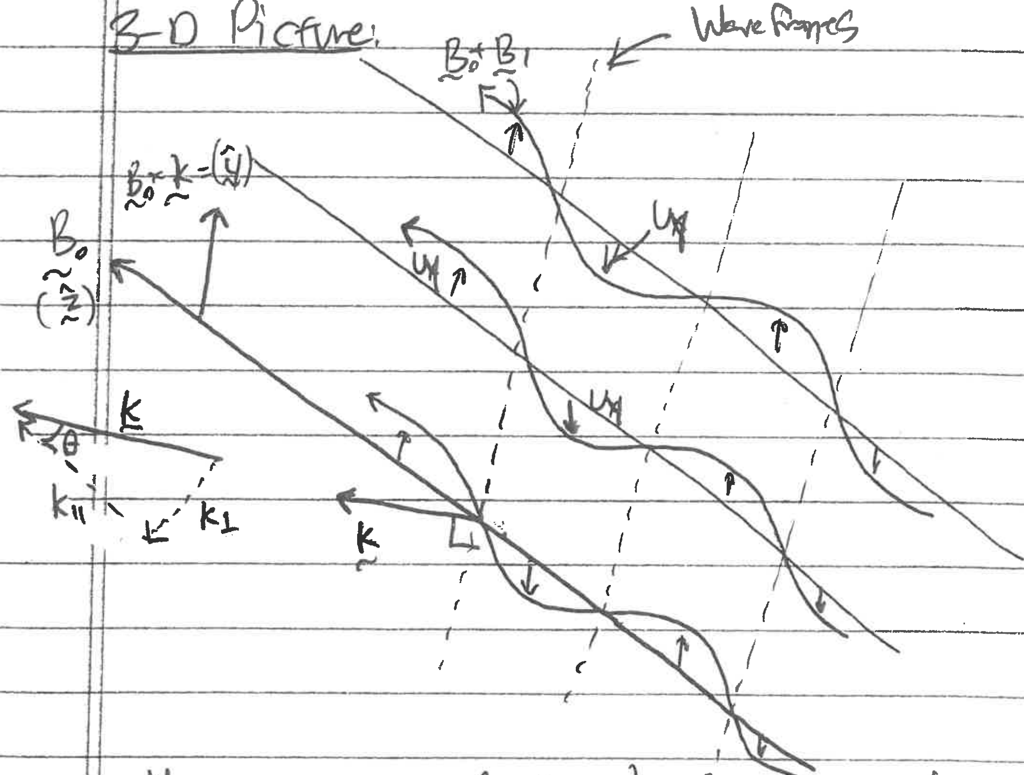
General MHD Dispersion Relation

3. Six Solutions: Three Waves, each with (+) & (−).

4. $\boxed{\text{Alfven Waves}}$ $\omega^2 = k^2\cos^2\theta\, V_A^2 \Rightarrow \boxed{\omega^2 = k_\parallel^2 V_A^2}$

a. Motion is in the $\hat{b} \times \hat{k}_\perp$ direction ($\hat{y}$ direction)

3-D Picture:



Polarization b. Motion is out of the plane defined by B_0 and k.

c. Restoring force is only magnetic tension

d. $k \cdot U_1 = 0 \Rightarrow$ Alfven wave is incompressible

Sometimes called the "Shear Alfven Wave"

Lecture #4 (Continued)

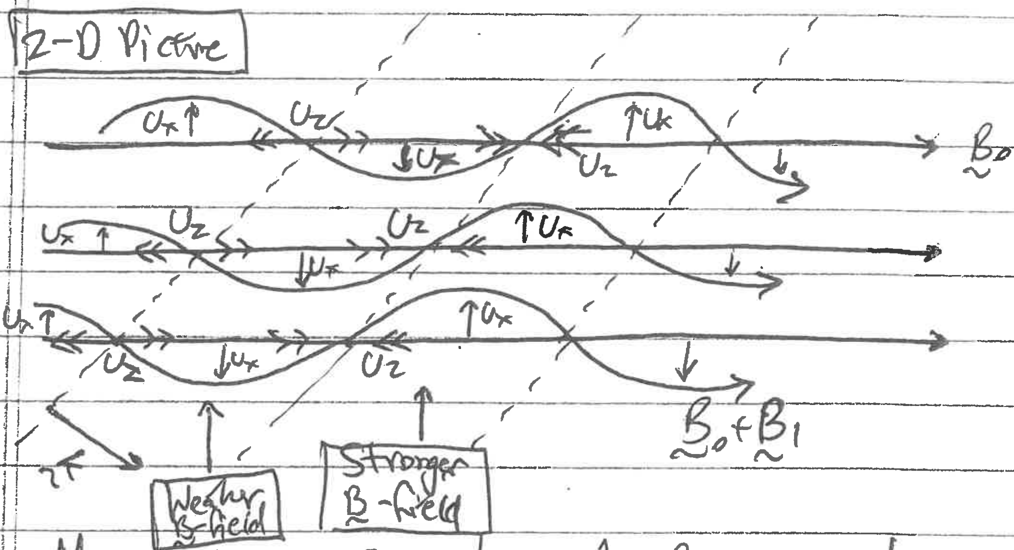
III. D. (Continued)

5. $\boxed{\text{Fast Waves}}$

phase speed $v_p = \frac{\omega}{k}$

$$\frac{\omega^2}{k^2} = \frac{1}{2}(c_s^2 + v_A^2) + \frac{1}{2}\sqrt{(c_s^2+v_A^2)^2 - 4c_s^2 v_A^2 \cos^2\theta}$$

2-D Picture



Polarization a. Motion is in the plane of $\underset{\sim}{B}_0$ and $\underset{\sim}{k}$

Has both $\hat{\underset{\sim}{b}} (= \hat{z})$ and $\underset{\sim}{k}_\perp (= \hat{x})$ components U_z & U_x

b. This wave is a mixture of Compressional (parallel) wave and transverse (perpendicular) wave.

- Restoring force: 1) Thermal and Magnetic pressure add together
  2) Bending of field lines - magnetic tension

c. Restoring force is strong because thermal & magnetic pressures add
$\Rightarrow$ Wave is fast.

d.1. For $\theta = 0$, $\omega^2 = \begin{cases} k^2 c_s^2 & c_s > v_A \quad \text{Sound Wave} \\ k^2 v_A^2 & v_A > c_s \quad \text{Alfven Wave} \end{cases}$

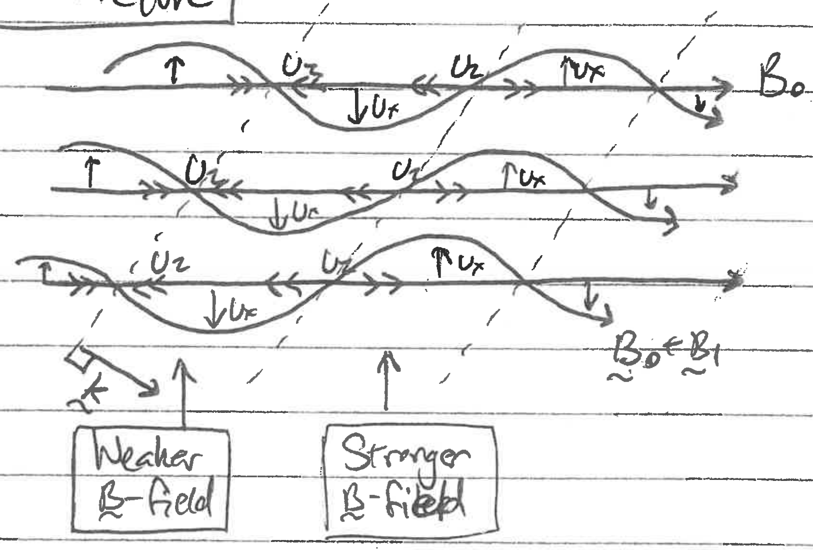
2. For $\theta = \frac{\pi}{2}$, $\omega^2 = k^2(c_s^2 + v_A^2)$ Magneto-acoustic wave

Lecture #4 (Continued)

III. D. (Continued)

6. Slow Waves $\quad \dfrac{\omega^2}{k^2} = \frac{1}{2}(c_s^2 + V_A^2) - \frac{1}{2}\sqrt{(c_s^2+V_A^2)^2 - 4c_s^2 V_A^2 \cos^2\theta}$

2-D Picture



Polarization a. Motion is in the plane of $\underset{\sim}{B}_0$ & $\underset{\sim}{k}$

Has both $\underset{\sim}{\hat{b}}\ (=\hat{z})$ and $\underset{\sim}{k}_\perp\ (=\hat{x})$ components U_z & U_x

b. Wave is a mixture of compressional (parallel) and tranverse (perpendicular) motions

- Restoring force: 1) Thermal pressure and Magnetic pressure oppose
  2) Magnetic tension due to bending of field lines

c. Restoring force is weak because thermal and magnetic pressures subtract

$\Rightarrow$ Wave is slow

d. For $\theta = 0$, $\quad \omega^2 = \begin{cases} k^2 c_s^2 & c_s^2 < V_A^2 \quad \text{Sound Wave} \\ k^2 V_A^2 & c_s^2 > V_A^2 \quad \text{Alfven Wave} \end{cases}$

e. For $\theta \to \frac{\pi}{2}$ $\quad \omega^2 \to 0$

Magnetic and thermal pressures subtract completely.