Lecture #9: Particle Motion in Slowly Varying $\underset{\sim}{E}$-Fields
Polarization Drift

I. Polarization Drift

A. Consider an Electric Field varying slowly in time $\underset{\sim}{E}(\tau)$ with a constant Magnetic Field $\underset{\sim}{B} = B_0 \hat{z}$.

1. NOTE: $\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{j} + \epsilon_0 \mu_0 \frac{\partial \underset{\sim}{E}}{\partial t}$, we assume $|\underset{\sim}{j}| \gg |\epsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t}|$, so $\frac{d B}{d t} \simeq 0$.

B. Mulitple Time Scale Analysis

1. $$\frac{d\underset{\sim}{v}}{dt} = \frac{q}{m}(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})$$

2. a. Take $\underset{\sim}{E}(\tau)$ varies only on slow timescale $\tau = \epsilon t$

b. $\underset{\sim}{v} = \underset{\sim}{v}_1(t) + \epsilon \underset{\sim}{v}_2(\tau) + \epsilon^2 \underset{\sim}{v}_3(\tau) + \ldots$

c. Also assume $\underset{\sim}{E}(\tau) \cdot \underset{\sim}{B} = 0$

3. As before,
$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau}$$

4. We'll take a small electric field such that the $E \times B$ drift velocity $v_E \ll v$, where v is Larmor orbit velocity.

Thus $$\frac{d\underset{\sim}{v}}{dt} = \frac{q}{m}(\epsilon \underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})$$

5. Subsitute expanded solution:

$$\frac{\partial}{\partial t}(\underset{\sim}{v}_1 + \cancelto{0}{\epsilon \underset{\sim}{v}_2} + \cancelto{0}{\epsilon^2 \underset{\sim}{v}_3}) + \epsilon \frac{\partial}{\partial \tau}(\cancelto{0}{\underset{\sim}{v}_1} + \epsilon \underset{\sim}{v}_2 + \epsilon^2 \underset{\sim}{v}_3) = \epsilon \frac{q}{m}\underset{\sim}{E}(\tau)$$
$$+ \frac{q}{m}(\underset{\sim}{v}_1 + \epsilon \underset{\sim}{v}_2 + \epsilon^2 \underset{\sim}{v}_3) \times \underset{\sim}{B}$$

a. Taking $\underset{\sim}{B} = B_0 \hat{\underset{\sim}{b}}$,

$$\frac{\partial \underset{\sim}{v}_1}{\partial t} + \epsilon^2 \frac{\partial \underset{\sim}{v}_2}{\partial \tau} + \epsilon^3 \frac{\partial \underset{\sim}{v}_3}{\partial \tau} = \epsilon \frac{q \underset{\sim}{E}(\tau)}{m} + \omega_c \underset{\sim}{v}_1 \times \hat{\underset{\sim}{b}} + \epsilon \omega_c \underset{\sim}{v}_2 \times \hat{\underset{\sim}{b}} + \epsilon^2 \omega_c \underset{\sim}{v}_3 \times \hat{\underset{\sim}{b}}$$

I. B. (Continued)

6. $\mathcal{O}(1)$: $\frac{\partial \underset{\sim}{v}_1}{\partial t} = \omega_c \underset{\sim}{v}_1 \times \hat{\underset{\sim}{b}}$

a. This is just the usual, fast timescale Larmor gyration about the magnetic field.

b. The general solution for this motion can be written

$$\boxed{\underset{\sim}{v}_1 = v_\perp \cos(\omega_c t + \phi)\hat{\underset{\sim}{e}}_1 - v_\perp \sin(\omega_c t + \phi)\hat{\underset{\sim}{e}}_2 + v_\parallel \hat{\underset{\sim}{b}}}$$

for a right-handed coordinate system s.t. $\hat{\underset{\sim}{e}}_1 \times \hat{\underset{\sim}{e}}_2 = \hat{\underset{\sim}{b}}$

7. $\mathcal{O}(\epsilon)$: $0 = \cancel{\epsilon}\frac{q}{m}\underset{\sim}{E}(\tau) + \cancel{\epsilon}\frac{qB_0}{m}\underset{\sim}{v}_2 \times \hat{\underset{\sim}{b}}$

a. This is just the slow timescale $\underset{\sim}{E} \times \underset{\sim}{B}$ drift.

b. Operating $\hat{\underset{\sim}{b}} \times$ on equation gives:

$$\hat{\underset{\sim}{b}} \times \underset{\sim}{E}(\tau) = -\frac{\cancel{q}B_0}{\cancel{m}}\hat{\underset{\sim}{b}} \times (\underset{\sim}{v}_2 \times \hat{\underset{\sim}{b}}) = -B_0\left(\underset{\sim}{v}_2(\hat{\underset{\sim}{b}}\cdot\hat{\underset{\sim}{b}}) - v_{2\parallel}\hat{\underset{\sim}{b}}\right)$$

or

$$\boxed{\underset{\sim}{v}_2 = v_{2\parallel}\hat{\underset{\sim}{b}} + \frac{\underset{\sim}{E}(\tau) \times \underset{\sim}{B}}{B_0^2}}$$

8. $\mathcal{O}(\epsilon^2)$: $\cancel{\epsilon^2}\frac{\partial \underset{\sim}{v}_2}{\partial \tau} = \cancel{\epsilon^2}\omega_c \underset{\sim}{v}_3 \times \hat{\underset{\sim}{b}}$

a. At this order, the solution $\underset{\sim}{v}_2$ is considered to be known.

Thus, $\frac{\partial \underset{\sim}{v}_2}{\partial \tau} = \cancel{\frac{\partial v_{2\parallel}}{\partial \tau}}^{\,0}\hat{\underset{\sim}{b}} + \frac{1}{B_0^2}\frac{\partial \underset{\sim}{E}}{\partial \tau} \times \underset{\sim}{B} = \frac{\frac{\partial \underset{\sim}{E}}{\partial \tau} \times \hat{\underset{\sim}{b}}}{B_0}$

b. $\frac{1}{B_0}\frac{\partial \underset{\sim}{E}}{\partial \tau} \times \hat{\underset{\sim}{b}} = \omega_c \underset{\sim}{v}_3 \times \hat{\underset{\sim}{b}}$

c. Take $\hat{\underset{\sim}{b}} \times$ this equation

$$\frac{1}{B_0}\hat{\underset{\sim}{b}} \times \left(\frac{\partial \underset{\sim}{E}}{\partial \tau} \times \hat{\underset{\sim}{b}}\right) = \frac{1}{B_0}\left[\frac{\partial \underset{\sim}{E}}{\partial \tau}(\hat{\underset{\sim}{b}}\cdot\hat{\underset{\sim}{b}}) - \hat{\underset{\sim}{b}}\cancel{\left(\hat{\underset{\sim}{b}}\cdot\frac{\partial \underset{\sim}{E}}{\partial \tau}\right)}^{\,0}\right] = \frac{1}{B_0}\frac{\partial \underset{\sim}{E}}{\partial \tau}$$

$$\omega_c \hat{\underset{\sim}{b}} \times (\underset{\sim}{v}_3 \times \hat{\underset{\sim}{b}}) = \omega_c\left[\underset{\sim}{v}_3(\hat{\underset{\sim}{b}}\cdot\hat{\underset{\sim}{b}}) - \hat{\underset{\sim}{b}}(\underset{\sim}{v}_3\cdot\hat{\underset{\sim}{b}})\right] = \omega_c\left(\underset{\sim}{v}_3 - v_{3\parallel}\hat{\underset{\sim}{b}}\right)$$

d. Thus

$$\boxed{\underset{\sim}{v}_3 = v_{3\parallel}\hat{\underset{\sim}{b}} + \frac{1}{\omega_c B_0}\frac{\partial \underset{\sim}{E}(\tau)}{\partial \tau}}$$

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9. Putting the full solution together: (Taking $v_{\parallel}$ $v_{2\parallel} = v_{3\parallel} = 0$)

$$\boxed{\underset{\sim}{v} = \underbrace{v_\perp\left(\cos(\omega_c t+\phi)\hat{\underset{\sim}{e}}_1 - \sin(\omega_c t+\phi)\hat{\underset{\sim}{e}}_2\right)}_{\text{Zeroth order Larmor Motion}} + \underbrace{\frac{\underset{\sim}{E}(t)\times\underset{\sim}{B}}{B^2}}_{\text{First-order } \underset{\sim}{E}\times\underset{\sim}{B} \text{ drift}} + \underbrace{\frac{1}{\omega_c B_0}\frac{d\underset{\sim}{E}}{dt}}_{\text{Second-order Polarization Drift}}}$$

C. Polarization Drift:

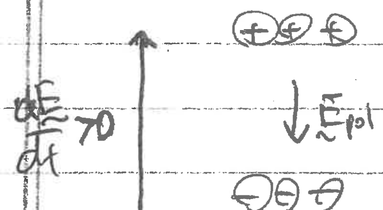
1. For slowly varying electric field $\underset{\sim}{E}(t)$ (slow with respect to the Larmor motion), we define the

Polarization Drift $\boxed{\underset{\sim}{v}_P \equiv \frac{1}{\omega_c B}\frac{d\underset{\sim}{E}}{dt}}$

2. Using $\omega_c = \frac{qB}{m}$, we have

$$\underset{\sim}{v}_P = \frac{m}{qB^2}\frac{d\underset{\sim}{E}}{dt}$$

a. Polarization drift is charge dependent
$\Rightarrow$ ions and electrons drift in opposite directions



b. Resulting polarization of plasma opposes increasing applied Electric Field.

c. Because $m_i \gg m_e$, ions dominate the polarization drift.

3. Polarization Current: $\underset{\sim}{j}_P = \sum_s q_s n_s \underset{\sim}{v}_P = \sum_s \frac{q_s n_s m_s}{q_s B^2}\frac{d\underset{\sim}{E}}{dt}$

a. $\underset{\sim}{j}_P = \sum_s \frac{n_s m_s}{B^2}\frac{d\underset{\sim}{E}}{dt}$

b. Mass dependence means ion contribute most to polarization current.

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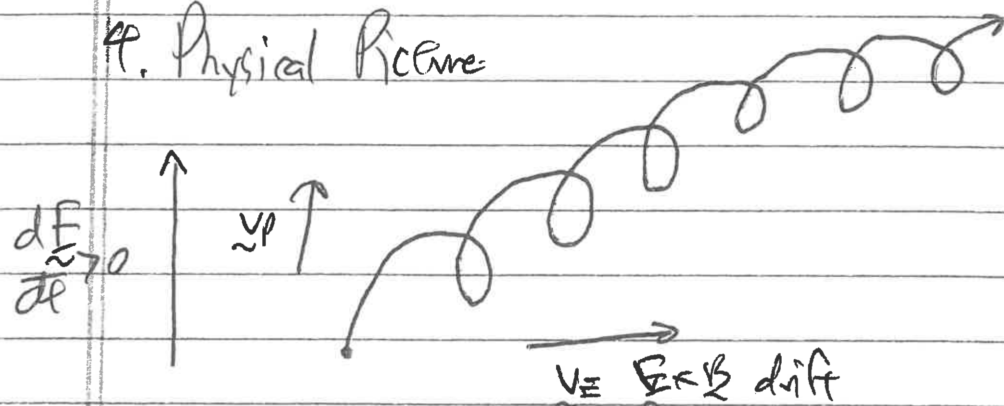
b. NOTE: $\underset{\sim}{E} \times \underset{\sim}{B}$ velocity is the same for both species, so it cancels, producing no net current.

$$\underset{\sim}{j}_E = \sum_s q_s n_s \underset{\sim}{v}_E = \sum_s q_s n_s \frac{\underset{\sim}{E} \times \underset{\sim}{B}}{B^2} = \frac{\underset{\sim}{E} \times \underset{\sim}{B}}{B^2} \underbrace{\cancel{\sum_s q_s n_s}}_{0 \text{ by quasineutrality}} = 0$$

4. Physical Picture



5. The Polarization Drift can lead to an increase in energy.

a. $\frac{d\mathcal{E}}{dt} = \underset{\sim}{v} \cdot \underset{\sim}{F} = \underset{\sim}{v} \cdot q(\underset{\sim}{E} + \cancel{\underset{\sim}{v} \times \underset{\sim}{B}}^{\,0}) = q \underset{\sim}{v} \cdot \underset{\sim}{E}$

b. $= q\Big[\underbrace{v_\perp \cos(\omega_c t + \phi)\, \underset{\sim}{E} \cdot \hat{\underset{\sim}{e}}_1 + v_\perp \sin(\omega_c t + \phi)\, \underset{\sim}{E} \cdot \hat{\underset{\sim}{e}}_2}_{\text{Average over Larmor orbit} \Rightarrow 0}$

$+ \cancel{\frac{\underset{\sim}{E}(t) \times \underset{\sim}{B}}{B^2} \cdot \underset{\sim}{E}}^{\,0} + \frac{1}{\omega_c B} \frac{d\underset{\sim}{E}}{dt} \cdot \underset{\sim}{E} \Big]$

c. Thus $\frac{d\mathcal{E}}{dt} = \frac{q}{\omega_c B} \frac{d}{dt}\left(\frac{E^2}{2}\right) = \frac{d}{dt}\left[\frac{1}{2} m \left(\frac{E^2}{B^2}\right)\right]$

d. Note: $|v_E|^2 = \frac{E^2}{B^2}$, so this can be written $\boxed{\frac{d}{dt}\left(\frac{1}{2} m v_E^2\right) = \frac{d\mathcal{E}}{dt}}$

e. **The Polarization Drift leads to the acceleration of particles to achieve the $\underset{\sim}{E} \times \underset{\sim}{B}$ drift velocity.**