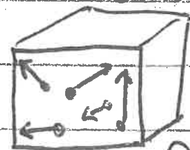
Lecture #(-1) Moment Equations

I. Review of Kinetic Description of Plasmas

A. Picture of a Kinetic Plasma

1. At any point in space, a kinetic plasma contains particles moving with a distribution of velocities



2. The distribution function $f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)$ describes a statistical measure of the plasma in a volume of ~~scale~~ x_0^3 such that

$x_0 \gg n^{-\frac{1}{3}}$ larger than particle spacing $\Rightarrow$ many particles

$x_0 \ll \lambda_D$ smaller than Debye length

3. $f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)$ is a number density in 6-D phase space $(\underset{\sim}{x}, \underset{\sim}{v})$

B. Evolution of distribution function

1. Along particle orbit, $\displaystyle \frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + \frac{d\underset{\sim}{x}}{dt} \cdot \frac{\partial f_s}{\partial \underset{\sim}{x}} + \frac{\partial \underset{\sim}{v}}{\partial t} \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}}$

2. The statistical ensemble average yields Plasma Kinetic Equation

$$\underbrace{\frac{\partial f_s}{\partial t} + \underset{\sim}{v} \cdot \nabla f_s + \frac{q_s}{m_s}\left(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}\right) \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}}}_{\text{Statistically smooth description of particles } f_s \text{, and fields } \underset{\sim}{E} \,\&\, \underset{\sim}{B} \text{ over volume } x_0^3} = \underbrace{\left(\frac{\partial f_s}{\partial t}\right)_{coll}}_{\text{Effect of discrete particles on smaller scales} \Rightarrow \text{collisions.}}$$

3. Neglect the effect of collisions yields Vlasov Equation

$$\frac{\partial f_s}{\partial t} + \underset{\sim}{v} \cdot \nabla f_s + \frac{q_s}{m_s}\left(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}\right) \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} = 0$$

I (Continued)

C. Moments of the Distribution Function

1. Summing over velocity space gives useful macroscopic quantities.

a. Density $n_s(\underset{\sim}{x},t) = \int d^3\underset{\sim}{v}\ f_s(\underset{\sim}{x},\underset{\sim}{v},t)$

b. Fluid Velocity $\underset{\sim}{U}_s(\underset{\sim}{x},t) = \frac{1}{n_s(\underset{\sim}{x},t)} \int d^3\underset{\sim}{v}\ \underset{\sim}{v}\ f_s(\underset{\sim}{x},\underset{\sim}{v},t)$

c. Kinetic Energy Density: $\mathcal{E}_s(\underset{\sim}{x},t) = \int d^3\underset{\sim}{v}\ \frac{1}{2} m |\underset{\sim}{v}|^2 f_s(\underset{\sim}{x},\underset{\sim}{v},t)$

d. Pressure Tensor: $\underset{\approx}{P}_s(\underset{\sim}{x},t) = \int d^3\underset{\sim}{v}\ m_s(\underset{\sim}{v}-\underset{\sim}{U}_s)(\underset{\sim}{v}-\underset{\sim}{U}_s) f_s(\underset{\sim}{x},\underset{\sim}{v},t)$

2. These are the velocity moments of the distribution function.

3. Taking moments of the kinetic distribution function gives "fluid" quantities, ~~or~~ observable quantities.

a. Ex: The temperature of a kinetic species s can be defined.

$$\mathcal{E}_s \equiv \frac{3}{2} n_s k T_s \quad \Rightarrow T_s \equiv \frac{2\mathcal{E}_s}{3 n_s k}$$

NOTE: From thermodynamics, $p_s = n_s k T_s$, so $\mathcal{E}_s = \frac{3}{2} p_s$, as expected.

II. From a Kinetic to a Fluid Description

A. Fluid Description:

1. Often, we don't care about detailed distribution of velocities.
2. Thus, all we want are velocity moments, n_s, $\underset{\sim}{U}_s$, $\mathcal{E}_s$, etc.
3. The velocity moments just define the dependent variables of a fluid description, which depend only on $\underset{\sim}{x}$ and t.

II. (Continued)

B. Comparison of Kinetic vs. Fluid Descriptions

Kinetic

1. $$\frac{\partial f_s}{\partial t} + \underset{\sim}{v}\cdot\nabla f_s + \frac{q_s}{m_s}(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{B})\cdot\frac{\partial f_s}{\partial \underset{\sim}{v}} = 0$$

Charge Density: $\rho = \sum_s q_s \int d^3\underset{\sim}{v}\ f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)$

Current Density: $\underset{\sim}{j} = \sum_s q_s \int d^3\underset{\sim}{v}\ \underset{\sim}{v}\ f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)$

$\nabla\cdot\underset{\sim}{E} = \frac{\rho}{\epsilon_0}$ $\qquad \nabla\cdot B = 0$

$\nabla\times E = -\frac{\partial B}{\partial t}$ $\qquad \nabla\times\underset{\sim}{B} = \mu_0\underset{\sim}{j} + \mu_0\epsilon_0\frac{\partial \underset{\sim}{E}}{\partial t}$

2. Everything depends on 6-D distribution function $f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)$

Fluid:

1. Moments: $n_s(\underset{\sim}{x}, t)$
$\underset{\sim}{U}_s(\underset{\sim}{x}, t)$
$\tilde{\varepsilon}_s(\underset{\sim}{x}, t)$
etc.

2. Evolution equations (to be derived next)

3. A number of fluid variables (moments), all of which only depend on $\underset{\sim}{x}$ and t.

C. When is a fluid description appropriate?

ANSWER: When the details of the velocity distribution do not significantly affect the evolution.

1. Ex: Fluid description is not valid when particles at certain velocity become resonant with fluctuating EM fields.

$\Rightarrow$ Landau Damping 

$\Rightarrow$ Cyclotron Damping 

2. Ex: Fluid description is appropriate for strongly collisional plasmas. Collisions drive towards Local Thermodynamic Equilibrium

$\Rightarrow$ Velocity distribution becomes Maxwellian.

I. (Continued)

D. Maxwellian (and Bi-Maxwellian) Plasmas:

1. Fluid equations are ~~an~~ good description of a plasma when the velocity distributions remain Maxwellian

2. Maxwellian Distribution $\boxed{f_{s_m}(\underset{\sim}{x},\underset{\sim}{v},t) = \frac{n_s(\underset{\sim}{x},t)}{\pi^{3/2} v_{ts}^3} e^{-\frac{m_s|\underset{\sim}{v}-\underset{\sim}{U}(\underset{\sim}{x},t)|^2}{2kT_s(\underset{\sim}{x},t)}}}$

a. Fluid is completely described by three fluid variables:

Density $n_s(\underset{\sim}{x},t)$

Fluid Velocity $\underset{\sim}{U}_s(\underset{\sim}{x},t)$

Temperature $T_s(\underset{\sim}{x},t)$

b. Here, as usual, the thermal velocity $v_{ts} = \sqrt{\frac{2kT_s}{m_s}}$

3. Zeroth Moment: Density

a. $\int d^3\underset{\sim}{v}\, f_{s_m}(\underset{\sim}{x},\underset{\sim}{v},t) = \int_{-\infty}^{\infty} dv_x\, dv_y\, dv_z \frac{n_s}{\pi^{3/2} v_{ts}^3} e^{-\frac{|\underset{\sim}{v}-\underset{\sim}{U}|^2}{v_{ts}^2}}$

b. NOTE: $|\underset{\sim}{v}-\underset{\sim}{U}|^2 = (v_x-U_x)^2 + (v_y-U_y)^2 + (v_z-U_z)^2$

c. We can split this into three integrals which are identical

$$\frac{1}{\pi^{1/2} v_{ts}} \int_{-\infty}^{\infty} dv_x\, e^{-\frac{(v_x-U_x)^2}{v_{ts}^2}} = \frac{1}{\pi^{1/2} v_{ts}} \underbrace{\int_{-\infty}^{\infty} v_{ts}\, dy\, e^{-y^2}}_{= v_{ts}\sqrt{\pi}} = \frac{\cancel{\sqrt{\pi}}\, \cancel{v_{ts}}}{\cancel{\pi^{1/2}}\, \cancel{v_{ts}}} = 1$$

$y = \frac{v_x - U_x}{v_{ts}} \quad dy = \frac{dv_x}{v_{ts}}$

d. Thus $\boxed{\int d^3\underset{\sim}{v}\, f_{s_m}(\underset{\sim}{x},\underset{\sim}{v},t) = n_s(\underset{\sim}{x},t)}$

II. D. (Continued)

4. First Moment: Fluid Velocity

$$\int d^3\underset{\sim}{v}\, \underset{\sim}{v} f_{s_m}(\underset{\sim}{x}, \underset{\sim}{v}, t) = n_s(\underset{\sim}{x}, t)\, \underset{\sim}{U}_s(\underset{\sim}{x}, t)$$

5. Second Moment: Energy

$$\int d^3\underset{\sim}{v}\, \frac{1}{2} m_s |\underset{\sim}{v}|^2 f_{s_m}(\underset{\sim}{x}, \underset{\sim}{v}, t) = \underbrace{\frac{3}{2} n_s k T_s}_{\text{Thermal Energy}} + \underbrace{\frac{1}{2} m_s n_s |\underset{\sim}{U}_s|^2}_{\text{Kinetic Energy of Fluid Flow}}$$

6. The presence of a Mean Magnetic Field $\underset{\sim}{B}_0$ often leads to an anisotropic distribution function

Bi-Maxwellian Distribution: $$\boxed{f_{s_{Bm}}(\underset{\sim}{x}, \underset{\sim}{v}, t) = \frac{n_s(\underset{\sim}{x}, t)}{\pi^{3/2} v_{t\perp s}^2 v_{t\parallel s}} e^{-\frac{m v_\perp^2}{2kT_{s\perp}} - \frac{m v_\parallel^2}{2kT_{s\parallel}}}}$$

where

$$v_{t\perp s} = \sqrt{\frac{2kT_{s\perp}}{m_s}} \quad \text{and} \quad v_{t\parallel s} = \sqrt{\frac{2kT_{s\parallel}}{m_s}}$$

a. This can lead to the $\boxed{\text{Chew–Goldberger Law, or Double Adiabatic, Equation of State}}$

III. Moment Equations:

Moments of the Plasma Kinetic Equation determine evolution of velocity moments.

A. Zeroth Moment: Notation $\int_{\underset{\sim}{v}} = \int d^3\underset{\sim}{v}$

$$\underbrace{\int_{\underset{\sim}{v}} \frac{\partial f_s}{\partial t}}_{①} + \underbrace{\int_{\underset{\sim}{v}} \underset{\sim}{v} \cdot \nabla f_s}_{②} + \underbrace{\int_{\underset{\sim}{v}} \frac{q_s}{m_s} (\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_s}_{③} = \underbrace{\int_{\underset{\sim}{v}} \left(\frac{\partial f_s}{\partial t}\right)_{coll}}_{④}$$

1. $① = \frac{\partial}{\partial t} \int_{\underset{\sim}{v}} f_s = \frac{\partial}{\partial t} n_s$

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III. A. (Continued)

2. ② $= \int_{\underset{\sim}{v}} \underset{\sim}{v} \cdot \nabla f_s = \nabla \cdot \int_{\underset{\sim}{v}} \underset{\sim}{v} f_s = \nabla \cdot (n_s \underset{\sim}{U_s})$

a. NOTE: $\underset{\sim}{x}$ and $\underset{\sim}{v}$ are independent variables, so $\nabla \cdot (\underset{\sim}{v} f_s) = \underset{\sim}{v} \cdot \nabla f_s + f_s \cancelto{0}{\nabla \cdot \underset{\sim}{v}}$

3. ③ $= \frac{q_s}{m_s} \int_{\underset{\sim}{v}} (\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} = \frac{q_s}{m_s} \int_{\underset{\sim}{v}} \nabla_{\underset{\sim}{v}} \cdot [(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) f_s]$

a. NOTE: $\underset{\sim}{E}(\underset{\sim}{x},t)$ & $\underset{\sim}{B}(\underset{\sim}{x},t)$ are independent of $\underset{\sim}{v}$: $\frac{\partial}{\partial \underset{\sim}{v}} \cdot (\underset{\sim}{E} f_s) = \underset{\sim}{E} \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} + f_s \cancelto{0}{\frac{\partial}{\partial \underset{\sim}{v}} \cdot \underset{\sim}{E}}$

b. $\frac{\partial}{\partial \underset{\sim}{v}} \cdot ((\underset{\sim}{v} \times \underset{\sim}{B}) f_s) = \underset{\sim}{v} \times \underset{\sim}{B} \cdot \frac{\partial f_s}{\partial \underset{\sim}{v}} + f_s \cancelto{0}{\frac{\partial}{\partial \underset{\sim}{v}} \cdot (\underset{\sim}{v} \times \underset{\sim}{B})}$

Since each component is like $\frac{\partial}{\partial v_x} (\underset{\sim}{v} \times \underset{\sim}{B})_x = \frac{\partial}{\partial v_x} (v_y B_z - v_z B_y) = 0$

c. Using NRL p.5 (28) Gauss's Thm: $\int_V d^3x \, \nabla \cdot \underset{\sim}{A} = \int_S d\underset{\sim}{S} \cdot \underset{\sim}{A}$

$\Rightarrow$ ③ $= \frac{q_s}{m_s} \int d\underset{\sim}{S}_v \cdot [(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \cdot f_s] = 0$

d. At $v \to \pm\infty$, $f_s \to 0$, so surface integral vanishes at infinity.

4. ④ $= \left(\frac{\partial}{\partial t} \int_{\underset{\sim}{v}} f_s \right)_{coll} = \left(\frac{\partial n_s}{\partial t} \right)_{coll} = 0$

Collisions don't create or destroy particles (number is conserved).

5. Thus, we are left with

$$\boxed{\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \underset{\sim}{U_s}) = 0}$$ Continuity Equation

a. Expresses conservation of particles

b. Integrate over a small volume V: $\int_V d^3x \frac{\partial n_s}{\partial t} = - \int_V d^3x \, \nabla \cdot (n_s \underset{\sim}{U_s})$

i) Once again, use Gauss's Thm on RHS.

ii) N = number of particles in V, $\Rightarrow \frac{\partial N}{\partial t} = - \int_S d\underset{\sim}{S} \cdot (n_s \underset{\sim}{U_s})$ Particles moving through surface.

~~Particles moving~~

III. B. First Moment: $\int_{\mathbf{v}} \mathbf{v}$ on Kinetic Equation

$$\underbrace{\int_{\mathbf{v}} \mathbf{v} \frac{\partial f_s}{\partial t}}_{1} + \underbrace{\int_{\mathbf{v}} \mathbf{v}\, \mathbf{v} \cdot \nabla f_s}_{2} + \underbrace{\int_{\mathbf{v}} \mathbf{v} \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}}}_{3} = \underbrace{\int_{\mathbf{v}} \mathbf{v} \left(\frac{\partial f_s}{\partial t}\right)_{coll}}_{4}$$

1. $(1) = \frac{\partial}{\partial t} \int_{\mathbf{v}} \mathbf{v} f_s = \frac{\partial}{\partial t}(n_s \mathbf{U}_s)$

2. This term involves Tensor mathematics.

$(2) = \int_{\mathbf{v}} \mathbf{v}\mathbf{v} \cdot \nabla f_s = \nabla \cdot \left(\int_{\mathbf{v}} \mathbf{v}\mathbf{v} f_s\right)$ since $\mathbf{x}$ & $\mathbf{v}$ are independent.

a. Use $\mathbf{v} = \mathbf{U}_s + \mathbf{v} - \mathbf{U}_s$ to get $\mathbf{v}\mathbf{v} = \underbrace{\mathbf{U}_s\mathbf{U}_s}_{a} + \underbrace{\mathbf{U}_s(\mathbf{v}-\mathbf{U}_s)}_{b} + \underbrace{(\mathbf{v}-\mathbf{U}_s)\mathbf{U}_s}_{c} + \underbrace{(\mathbf{v}-\mathbf{U}_s)(\mathbf{v}-\mathbf{U}_s)}_{d}$

b. $(a) = \int_{\mathbf{v}} \mathbf{U}_s\mathbf{U}_s f_s = n_s \mathbf{U}_s\mathbf{U}_s$

c. $(b) = \int_{\mathbf{v}} \mathbf{U}_s(\mathbf{v}-\mathbf{U}_s) f_s = \mathbf{U}_s \int \mathbf{v} f_s - \mathbf{U}_s\mathbf{U}_s n_s = \mathbf{U}_s\mathbf{U}_s n_s - \mathbf{U}_s\mathbf{U}_s n_s = 0$

d. (c) same as $(b) = 0$

e. (d) Using $\mathbf{P}_s = \int_{\mathbf{v}} m_s (\mathbf{v}-\mathbf{U}_s)(\mathbf{v}-\mathbf{U}_s) f_s$, $(d) = \frac{\mathbf{P}_s}{m}$

f. Thus $(2) = \nabla \cdot (n_s \mathbf{U}_s\mathbf{U}_s) + \frac{1}{m_s}\nabla \cdot \mathbf{P}_s$

3. $(3) = \frac{q_s}{m_s} \int_{\mathbf{v}} \mathbf{v} \frac{\partial}{\partial \mathbf{v}} \cdot \left[(\mathbf{E} + \mathbf{v}\times\mathbf{B}) f_s\right]$ using same trick as Zeroth Moment.

a. Nine integrals for each pairing of v_x, v_y, v_z & $\frac{\partial}{\partial v_x}, \frac{\partial}{\partial v_y}, \frac{\partial}{\partial v_z}$

i) Six have $v_i \frac{\partial}{\partial v_j}$ where $i \neq j$, Thus are similar to

$$\int dv_x \int dv_y\, v_y \int dv_z \frac{\partial}{\partial v_z}\left[(\mathbf{E} + \mathbf{v}\times\mathbf{B}) f_s\right]$$

ii) v_z integral gives $(\mathbf{E} + \mathbf{v}\times\mathbf{B}) f_s \Big|_{-\infty}^{\infty} = 0$ as $v \to \infty$ since $f_s \to 0$.

iii) Three have $v_i \frac{\partial}{\partial v_i} \Rightarrow \int dv_x \int dv_y \int dv_z\, v_z \frac{\partial}{\partial v_z}\left[(\mathbf{E} + \mathbf{v}\times\mathbf{B}) f_s\right]$

II. B3. (Continued)

$$\int dv_z\, v_z \frac{\partial}{\partial v_z}\left[(\underset{\sim}{E}+\underset{\sim}{v}\times\underset{\sim}{B})f_s\right] = \cancel{v_z(\underset{\sim}{E}+\underset{\sim}{v}\times\underset{\sim}{B})f_s\Big|_{-\infty}^{\infty}}^{\,0} - \int_{-\infty}^{\infty}(\underset{\sim}{E}+\underset{\sim}{v}\times\underset{\sim}{B})f_s\, dv_z$$

$u = v_z \qquad dv = \frac{\partial}{\partial v_z}[(\underset{\sim}{E}+\underset{\sim}{v}\times\underset{\sim}{B})f_s]dv_z$

$du = dv_z \qquad v = (\underset{\sim}{E}+\underset{\sim}{v}\times\underset{\sim}{B})f_s$

b. We get $③ = -\frac{n_s q_s}{m_s}(\underset{\sim}{E}+\underset{\sim}{U_s}\times\underset{\sim}{B})$

4. $④ = \frac{1}{m_s}\int_{\underset{\sim}{v}} m_s \underset{\sim}{v}\left(\frac{\partial f_s}{\partial t}\right)_{coll} = \frac{1}{m_s}\left(\frac{\partial}{\partial t}\int_{\underset{\sim}{v}} m_s \underset{\sim}{v} f_s\right)_{coll} = \frac{1}{m_s}\left(\frac{\partial\, n_s m_s \underset{\sim}{U_s}}{\partial t}\right)_{coll}$

a. This represents a DRAG FORCE due to collisions between species

b. Same species collisions conserve momentum, so produce no drag.

5. Thus,

$$\frac{\partial}{\partial t}(n_s\underset{\sim}{U_s}) + \nabla\cdot(n_s\underset{\sim}{U_s}\underset{\sim}{U_s}) = -\frac{1}{m_s}\nabla\cdot\underset{\approx}{P_s} + \frac{n_s q_s}{m_s}(\underset{\sim}{E}+\underset{\sim}{U_s}\times\underset{\sim}{B}) + \underset{\text{collisional Drag}}{(\underset{\sim}{F_{Ds}})}$$

a. NOTE: LHS can be written, using tensor identity $\nabla\cdot(\underset{\sim}{A}\underset{\sim}{B}) = (\nabla\cdot\underset{\sim}{A})\underset{\sim}{B} + (\underset{\sim}{A}\cdot\nabla)\underset{\sim}{B}$

$$n_s\frac{\partial \underset{\sim}{U_s}}{\partial t} + \underbrace{\underset{\sim}{U_s}\frac{\partial n_s}{\partial t} + \underset{\sim}{U_s}\,\nabla\cdot(n_s\underset{\sim}{U_s})}_{=\underset{\sim}{U_s}\left[\cancel{\frac{\partial n_s}{\partial t}+\nabla\cdot(n_s\underset{\sim}{U_s})}^{\,0}\right]\ \text{Continuity eq.}} + n_s\underset{\sim}{U_s}\cdot\nabla\underset{\sim}{U_s}$$

6. Multiplying by m_s, we get

Momentum Equation

$$\boxed{n_s m_s\frac{\partial \underset{\sim}{U_s}}{\partial t} + n_s m_s \underset{\sim}{U_s}\cdot\nabla\underset{\sim}{U_s} = -\nabla\cdot\underset{\approx}{P_s} + n_s q_s(\underset{\sim}{E}+\underset{\sim}{U_s}\times\underset{\sim}{B}) + \underset{\sim}{F_{Ds}}}$$

C. Second Moment: Taking $\int_v \frac{1}{2}m_s|\underset{\sim}{v}|^2$ of Kinetic Equation gives,

$$\boxed{\frac{\partial \mathcal{E}_s}{\partial t} + \nabla\cdot\underset{\sim}{Q_s} - \underset{\sim}{E}\cdot\underset{\sim}{j_s} = \left(\frac{\partial\mathcal{E}_s}{\partial t}\right)_{coll}}$$

Energy Equation

II. C. (Continued)

1. Here $\underset{\sim}{Q}_s$ is a heat flux

Heat Flux $\boxed{\underset{\sim}{Q}_s = \int d^3v\, \frac{1}{2} m_s |\underset{\sim}{v}|^2 \underset{\sim}{v}\, f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)}$ (Third Velocity Moment)

2. $\underset{\sim}{E} \cdot \underset{\sim}{j}_s$ represents Joule heating of species s

3. $\left(\frac{\partial \mathcal{E}_s}{\partial t}\right)_{coll}$ is collisional heating due to collisions with other species.

a. Same-species conserve energy within species.

D. Closure Problem:

1. The evolution equation for the n^{th} moment involves the $(n+1)^{st}$ moment.

a. $$\underbrace{\frac{\partial n_s}{\partial t}}_{\text{(zeroth)}} + \nabla \cdot \underbrace{(n_s \underset{\sim}{U}_s)}_{\text{(first)}} = 0$$

b. $$n_s m_s \left[\underbrace{\frac{\partial \underset{\sim}{U}_s}{\partial t}}_{\text{(first)}} + \underset{\sim}{U}_s \cdot \nabla \underset{\sim}{U}_s\right] = -\nabla \cdot \underbrace{\underset{\approx}{P}_s}_{\text{(second)}} + n_s q_s (\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) + \underset{\sim}{F}_{Ds}$$

2. For N moment equations, a closed system can only be specified if the $(N+1)^{st}$ moment is related to the first N moments.

Ex: We may specify an equation of state relate pressure in terms of density.

Adiabatic Equation of State $\qquad P_s n_s^{-\gamma} = \text{constant}$

$$\Rightarrow \frac{d}{dt}\left(\frac{P}{n^\gamma}\right) = 0$$