ASTR:7830 Homework #5

Suggested Reading: Read RLS Chapter 10 (p.315–343)

Due at the beginning of class, Friday, March 8, 2024.

1. Magnetic Field Due to a Ring of Current
   Consider a circular current loop of radius R_0 centered about the origin of a cylindrical coordinate system with current I_0 in the $-\hat{\phi}$ direction. The current density for this current loop may be expressed as
   $$\mathbf{j}(\mathbf{r}) = -I_0 \delta(r - R_0)\delta(z)\hat{\phi}.$$
   (a) Use the Biot-Savart Law to compute the magnetic field at the origin $\mathbf{r} = 0$ due to this current loop. Do not forget to include the vector direction in the answer for $\mathbf{B}(0)$.
   (b) The dipole moment for this current loop is $\boldsymbol{\mu} = -I_0 A\hat{\mathbf{z}}$, where the loop encircles the area A. Use the Biot-Savart Law to compute the far-field limit of the magnetic field due to a dipole at a position in the plane of the loop far away from the loop, $r \gg R_0$,
   $$\mathbf{B}(r, \phi, 0) = \frac{-\mu_0}{4\pi}\frac{\boldsymbol{\mu}}{r^3}$$

2. Radius of the Plasmasphere
   For cold plasma that has evaporated from the ionosphere, the convection of the plasma in the inner magnetosphere is governed by a balance of the convection electrostatic potential and the corotation electrostatic potential,
   $$\Phi_{tot} = -E_c r \sin\phi - \frac{\Omega_E B_E R_E^3}{r}.$$
   The stagnation point in the convective flow occurs where the electric field due to this potential is zero, $\mathbf{E} = -\nabla\Phi_{tot} = 0$.
   (a) Compute the formula for radius of the stagnation point in terms of Ω_E, B_E, R_E, and the convective electric field E_c. In which direction is the stagnation point from the Earth, noon, dusk, midnight, or dawn?
   (b) For a convective electric field of magnitude $E_c = 2$ mV/m, compute the L value of the stagnation point, which gives an estimate of the radius of the plasmapause, the outer boundary of the plasmasphere.

3. Partial Ring Current
   For hot plasma with energies $W \gtrsim 10$ keV, the convection of the plasma is governed by a balance of the convection electrostatic potential and the effective potential associated with the azimuthal drift due to the ∇B and curvature drifts,
   $$\Phi_{tot} = -E_c r \sin\phi + \frac{\mu B_E R_E^3}{q r^3},$$
   where the magnetic moment of the charged particle is $\mu = m v_\perp^2 / 2B$ and, for simplicity, we have assumed equatorial trapped particles with a pitch angle of $\alpha = 90°$.
   (a) Compute the formula for radius of the stagnation point in terms of the perpendicular kinetic energy $W_\perp = m v_\perp^2/2$ and the convective electric field E_c. In which direction is the stagnation point from the Earth, noon, dusk, midnight, or dawn?
   HINT: When taking the gradient of the potential, the magnetic moment—which is the first adiabatic invariant—is constant.
   (b) For particles with energy $W_\perp = 20$ keV and a convective electric field of magnitude $E_c = 1$ mV/m, compute the L value of the stagnation point. This gives an estimation of the radius at which the ring current transitions to the partial ring current for particles of this energy.