ASTR:7830 Homework #6

Suggested Reading: Read RLS16 Chapter 13, Sec 13.3–13.3.7 (p.398–415)
Read RLS16 Chapter 4 (p.94–143)

Due at the beginning of class, Friday, March 29, 2024.

1. Waves in a Cold, Unmagnetized Plasma
   Ionospheric sounding is based on the property that light waves cannot propagate in a plasma if the wave frequency is below the plasma frequency. Here, we will derive the linear dispersion relation for electromagnetic waves in a cold, unmagnetized plasma.

   Beginning with the moment equations (Lecture #4), we apply the cold plasma approximation $v_{te} \ll \omega/k$ so that we may close the set of equations by setting the pressure tensor to zero. Assuming that the singly charged ions are immobile and provide a neutralizing background ($\mathbf{U}_i = 0$, $q_i = -q_e$, $n_{i0} = n_{e0} = n_0$), we are left with the electron continuity and momentum equations and Maxwell's equations,

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{U}_e) = 0$$

$$m_e n_e \left(\frac{\partial \mathbf{U}_e}{\partial t} + \mathbf{U}_e \cdot \nabla \mathbf{U}_e \right) = -e n_e \left(\mathbf{E} + \mathbf{U}_e \times \mathbf{B} \right)$$

$$\nabla \cdot \mathbf{E} = \frac{\sum_s n_s q_s}{\epsilon_0}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 (\sum_s n_s q_s \mathbf{U}_s) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

   (a) Linearization: Assume the following ordering,

$$\begin{array}{llll} n_i = n_0, & & n_e = n_0 & + \epsilon n_{e1} \\ \mathbf{U}_i = 0, & & \mathbf{U}_e = & \epsilon \mathbf{U}_{e1} \\ \mathbf{B} = & \epsilon \mathbf{B}_1, & \mathbf{E} = & \epsilon \mathbf{E}_1 \; . \end{array}$$

   Compute the linearized electron continuity and momentum equations and the linearized Maxwell's equations.
   HINT: Eliminate the lowest order of charge density fluctuations using quasineutrality, $\rho_{q0} = \sum_s n_{s0} q_s = q_i n_0 + q_e n_0 = 0$.

   (b) Write down the linearized equations above after Fourier transformation in time and space.

   (c) Eliminate $\mathbf{U}_{e1}$ by the using the electron momentum equation to substitute into the Ampere-Maxwell Law. Simplify the resulting equation using the defintion of the electron plasma frequency, $\omega_{pe}^2 = n_0 q_e^2 / (\epsilon_0 m_e)$ and $\mu_0 \epsilon_0 = 1/c^2$.

   (d) Eliminate $\mathbf{B}_1$ in the equation above by using Faraday's Law.

   (e) Assuming a wavevector $\mathbf{k} = k\hat{\mathbf{z}}$, write the problem as a matrix equation of the form

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} E_{x1} \\ E_{y1} \\ E_{z1} \end{pmatrix} = 0$$

   in terms of ω, k, ω_{pe}, and c.

   (f) Determine the dispersion relation $D(\omega, \mathbf{k}) = 0$ by setting the determinant of the 3×3 matrix M equal to zero, $|M| = 0$.

   (g) Write down the possible solutions to this dispersion relation.

2. Ionospheric Sounding
A rough model of the electron density n_e vs. altitude z in the ionosphere is given by

$$\log_{10}\left(\frac{n_e(z)}{n_0}\right) = 9\left(\frac{z - H_0}{H_I}\right)\exp - \left(\frac{z - H_0}{H_I}\right)$$

where $n_0 = 10^3$ cm^{-3}, $H_0 = 60$ km, and $H_I = 190$ km. The model is valid for altitudes $H_0 < z < 5H_I$.

(a) Find the minimum wave linear frequency $f = \omega/2\pi$ that can be used to communicate with a satellite in geosynchronous orbit. Give your answer in units of MHz.

(b) Compute the altitude at which a radio wave of frequency $f = 5$ MHz launched from the ground will reflect.

(c) In instead, the radio wave of frequency $f = 5$ MHz was launched down from a spacecraft at an altitude above the surface of $z = 6000$ km, at what altitude would the wave reflect?