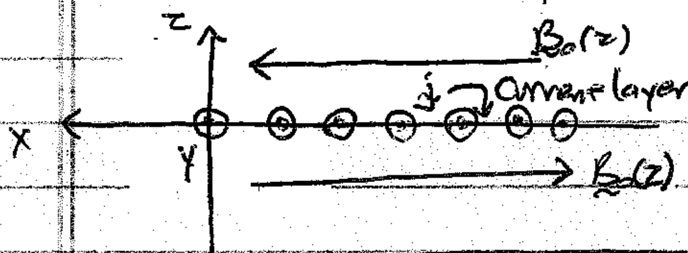
Howes ①

Lecture #14: Magnetic Reconnection, Sweet-Parker Model, Petschek Model

I. Steady State Magnetic Reconnection

A. Resistive Magnetic Diffusion with no plasma flow, $\underset{\sim}{U}=0$

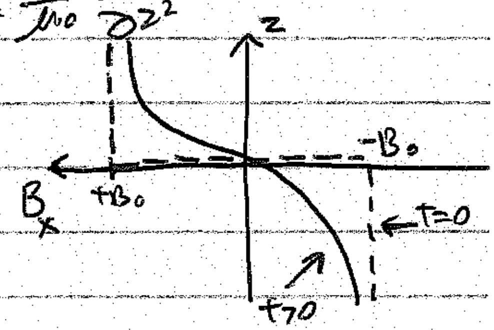
1. When the plasma is not flowing, antiparallel magnetic fields will evolve according to the diffusion equation,



$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times \cancel{(\underset{\sim}{U} \times \underset{\sim}{B})}^{0} + \frac{\eta}{\mu_0}\nabla^2 \underset{\sim}{B}$$

$$\Rightarrow \frac{\partial B_x}{\partial t} = \frac{\eta}{\mu_0}\frac{\partial^2 B_x}{\partial z^2}$$

2. Solution: $\boxed{B_x(z,t) = B_0 \,\mathrm{erf}\left[\left(\frac{\mu_0}{2\eta t}\right)^{\frac{1}{2}} z\right]}$



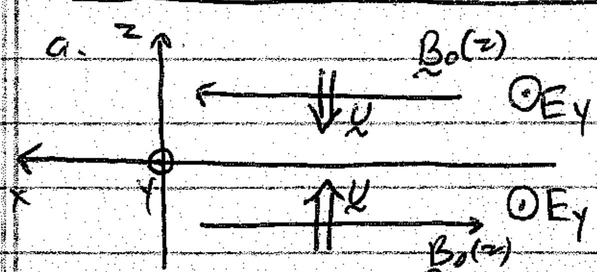
3. a. Since Magnetic energy $\int d^3r \frac{B^2}{2\mu_0}$ is lost, we cannot achieve a steady state.

   b. To obtain a steady state, we need an equal flow of magnetic energy towards the current layer.

B. Steady State Model of Magnetic Reconnection:

1. Inflow due to $\underset{\sim}{E} \times \underset{\sim}{B}$ drift: 2-D model, $\frac{\partial}{\partial y}=0$

a.



b. To obtain an inflow, we require $\underset{\sim}{E} = E_y \hat{y}$ both above and below the current sheet.

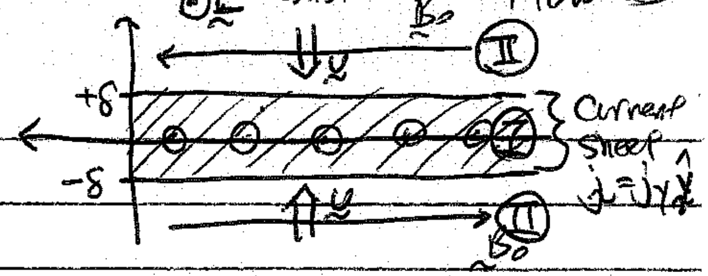
c. Faraday's Law: $\frac{\partial \underset{\sim}{B}}{\partial t} = -\nabla \times \underset{\sim}{E}$

i) In steady state $\frac{\partial}{\partial t}=0 \Rightarrow -\nabla \times \underset{\sim}{E} = 0$

ii) $\hat{x}$-component: $-\cancel{\frac{\partial E_z}{\partial y}}^{0} + \frac{\partial E_y}{\partial z} = 0 \Rightarrow \frac{\partial E_y}{\partial z} = 0 \Rightarrow \boxed{E_y \text{ is constant with } z!}$

I.B. (Continued)

2. Ohm's Law in Two Regions:



a. Ohm's Law: (Lect #5 I.B.6.)

$$\mathbf{E} = -\mathbf{U}\times\mathbf{B} + \eta \mathbf{j}$$

b. In Region (II) (away from current sheet), $\eta \mathbf{j}$ is negligible

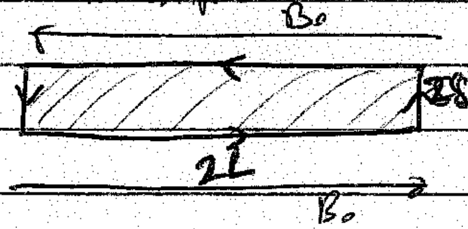
$$\Rightarrow \mathbf{E} = -\mathbf{U}\times\mathbf{B}_0 \Rightarrow \boxed{E_y = UB_0} \quad (1)$$

c. In Region (I) (current sheet), $\mathbf{B}_0 \to 0$, so $\mathbf{U}\times\mathbf{B}_0$ is negligible

$$\mathbf{E} = \eta \mathbf{j} \Rightarrow \boxed{E_y = \eta j_y} \quad (2)$$

3. Ampere's Law: $\nabla\times\mathbf{B} = \mu_0 \mathbf{j}$

a. We can compute this with an Amperian loop over current sheet



b. $\int d\mathbf{A}\cdot(\nabla\times\mathbf{B}) = \oint d\mathbf{l}\cdot\mathbf{B} = B_0(2L) + B_0(2L) = 4B_0L$

c. $\int d\mathbf{A}\cdot\mu_0\mathbf{j} = \mu_0(2L)(2\delta)j_y$

d. Thus $4B_0L = 4\mu_0 L\delta j_y \Rightarrow \boxed{j_y = \frac{B_0}{\mu_0\delta}} \quad (3)$

4. Compute Current Sheet Thickness as a function of inflow velocity U.

a. Using (1) & (2) to eliminate E_y and (3) to eliminate j_y, we obtain

$$\boxed{\delta = \frac{\eta}{\mu_0 U}} \quad (4)$$ Current sheet thickness for inflow U

b. Current sheet thickness adjusts to provide sufficient dissipation to balance inflow of magnetic energy.

c. For current sheet, $R_{eM} = \frac{\mu_0 L V_0}{\eta} = \frac{\mu_0\left(\frac{\eta}{\mu_0 U}\right)(U)}{\eta} = 1 \Leftarrow$ $\boxed{R_{eM} = 1 \text{ at edge of current sheet}}$

I. B. (Continued)

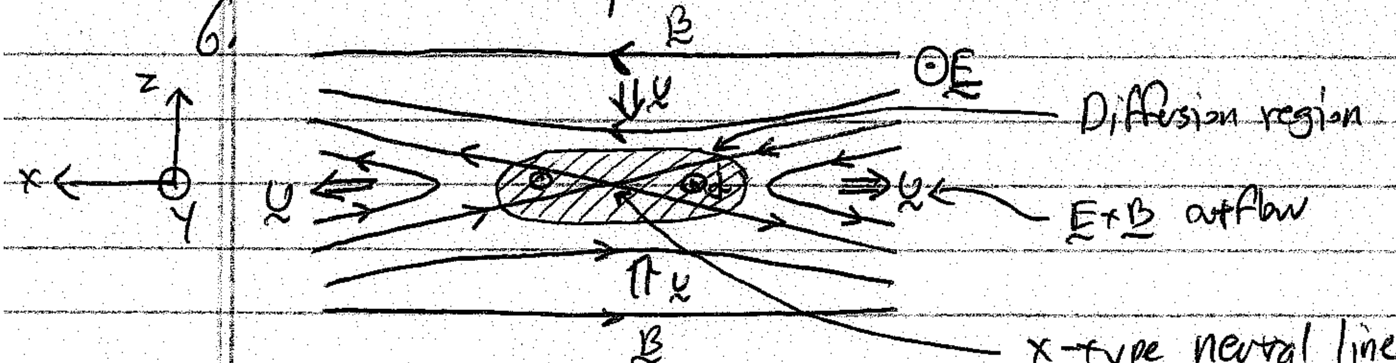
5. <u>Mass Pile-up</u>: Problem with this model!

a. As the anti-parallel magnetic field flows together at the current sheet magnetic energy is annihilated
(<u>NOTE</u>: Net magnetic flux in a symmetric system is zero, so net magnetic flux is conserved in this process).

b. But, since plasma is frozen to the magnetic field, plasma converges at $z=0$ as well $\Rightarrow$ But mass is not annihilated, so we will pile up mass at $z=0$ $\Rightarrow$ <u>Not a steady state</u>!

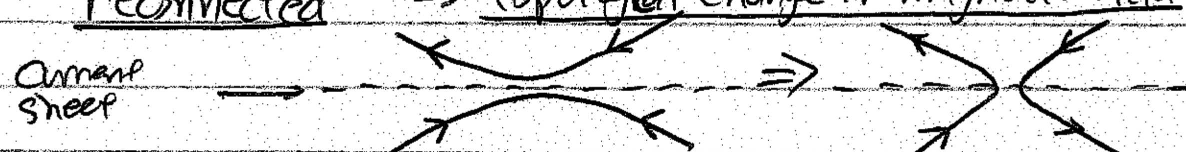
c. To solve this problem, plasma needs to escape the current sheet, and so we introduce variation in another dimension (z) to allow the plasma to flow out!

6.



a. Since $\nabla \times \underset{\sim}{E} = 0$ in steady state still, E_y = constant (as before

7. At x-line, magnetic field lines are broken and "<u>reconnected</u>" $\Rightarrow$ <u>topological change of magnetic field</u>



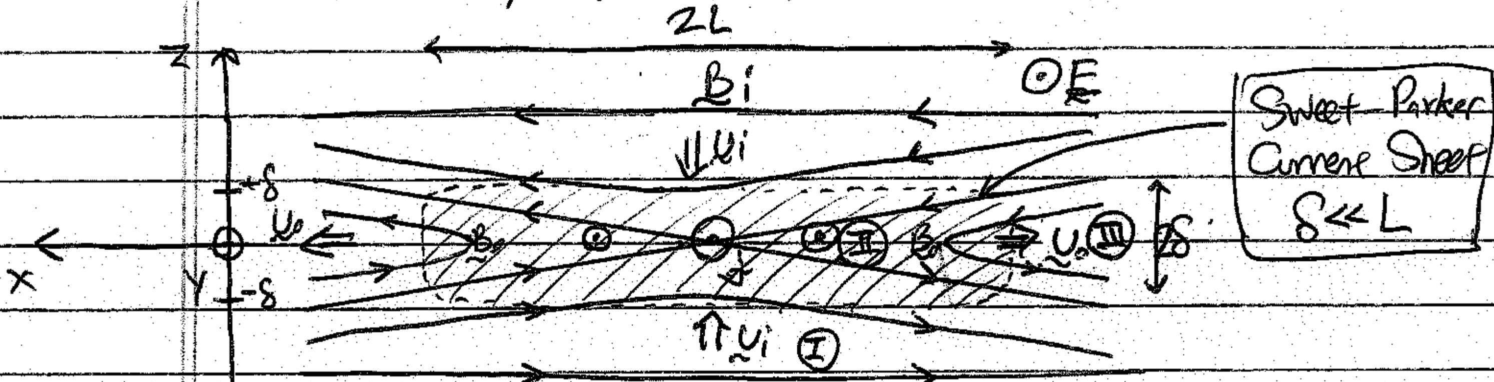
a. This reconnection enables plasma to flow along field lines from one side of current sheet to the other

b. <u>Example</u>: In reconnection at magnetopause, plasma of magnetosheath origin can mix with plasma of magnetosphere origin.

II. Sweet-Parker Reconnection:

A. General Setup:

1. 2-D System, $\frac{\partial}{\partial y} = 0$, Symmetrical inflow and outflow.



2. (I): Inflow Region: $\underline{U}_i$, $\underline{B}_i$

   (II): ~~Diffusion Region:~~

   (III): Outflow Region: $\underline{U}_0$, $\underline{B}_0$

3. Assumption incompressible flow, $\rho_i = \rho_0 = \rho = \text{constant}$

B. Conservation of Mass:

1. Consider the flow into and out of diffusion region.
   a. Assumption an out-of-plane width w in $\hat{y}$ direction

2a. Mass flow into diffusion region: $\frac{\text{Mass}}{\text{time}} = \rho U_i (2L) w$

b. Mass flow out of diffusion region: $\frac{\text{Mass}}{\text{time}} = \rho U_0 (2\delta) w$

3. In steady-state, inflow and outflow rates must be the same!

$$\rho U_i 2 L w = \rho U_0 2 \delta w \Rightarrow \boxed{\frac{U_i}{U_0} = \frac{\delta}{L}} \quad (5)$$

Since $\frac{\delta}{L} \ll 1$, $U_i \ll U_0$!

II. (Continued)

C. Conservation of Energy

1. Magnetic and Kinetic Energy flowing into Sweet-Parker current sheet is equal to that flowing out.

2. Electromagnetic Energy inflow rate per unit area is Poynting Flux

a. $|\underset{\sim}{S}| = \frac{1}{\mu_0}|\underset{\sim}{E} \times \underset{\sim}{B}| = \frac{E_y B_i}{\mu_0}$

b. As before, $\nabla \times \underset{\sim}{E} = 0$ in steady state, so $E_y = \text{const.}$

$$E_y = U_i B_i$$

c. Thus $S_i = \frac{U_i B_i^2}{\mu_0}$. Similarly, $S_o = \frac{U_o B_o^2}{\mu_0}$

3. Kinetic Energy flow rate per unit area is

$\frac{1}{2}\rho U_i^3$ inflow, $\frac{1}{2}\rho U_o^3$ outflow

4. Total Energy Conservation: Balance (EM + KE)(Area)

a. $$\left(\frac{U_i B_i^2}{\mu_0} + \frac{1}{2}\rho U_i^3\right)(2Lw) = \left(\frac{U_o B_o^2}{\mu_0} + \frac{1}{2}\rho U_o^3\right)(2\delta w)$$

b. $$\frac{U_i}{U_o}\left[\frac{B_i^2}{\mu_0 \rho} + \frac{U_i^2}{2}\right](2Lw) = \left[\frac{B_o^2}{\mu_0 \rho} + \frac{U_o^2}{2}\right](2\delta w)$$

c. Using $\frac{U_i}{U_o} = \frac{\delta}{L}$ and $V_{A_i}^2 = \frac{B_i^2}{\mu_0 \rho}$ and $V_{A_o}^2 = \frac{B_o^2}{\mu_0 \rho}$, we obtain

$$\boxed{V_{A_i}^2 + \frac{U_i^2}{2} = V_{A_o}^2 + \frac{U_o^2}{2}}$$

5. For the Sweet-Parker current sheet with $\delta \ll L$,

a. $U_o = \frac{L}{\delta} U_i \gg U_i$, so we may drop U_i^2

b. If a significant amount of magnetic energy is dissipated,

$V_{A_i}^2 \gg V_{A_o}^2$, so we may drop $V_{A_o}^2$

II. C. (Continued)

6. Thus $\boxed{U_o^2 = 2 V_{A_i}^2}$ (6) Conversion of magnetic to kinetic energy.

⇒ Outflow rate is about equal to Alfvén velocity in inflow region

D. Reconnection Rate Scaling with Lundquist Number, S

1. Matching Diffusion region (II) with Inflow Region (I) yields same criterion for current sheet thickness (see I.B.4).

$$\delta = \frac{\eta}{\mu_0 U_i} \quad (4)$$

2. To determine the inflow rate relative to Alfven speed, $\frac{U_i}{V_{A_i}}$, Use (4), (5), and (6) to eliminate U_o and δ:

a. $$\frac{U_i}{V_{A_i}} = \frac{\delta}{L}\underbrace{\left(\frac{U_o}{V_{A_i}}\right)}_{=\sqrt{2}} = \sqrt{2}\,\frac{\left(\frac{\eta}{\mu_0 U_i}\right)}{L} \Longrightarrow \frac{U_i^2}{V_{A_i}} = \sqrt{2}\,\frac{\eta}{\mu_0 L}$$

b. Divide by V_{A_i} and take root:

$$\frac{U_i}{V_{A_i}} = \frac{2^{\frac{1}{4}}}{\left[\frac{\mu_0 L V_{A_i}}{\eta}\right]^{\frac{1}{2}}} = 2^{\frac{1}{4}} S^{-\frac{1}{2}}$$

3. Define Lundquist Number: $\boxed{S \equiv \frac{\mu_0 L V_{A_i}}{\eta}}$

This is R_{EM} with the velocity $V_o = V_{A_i}$.

4. $\boxed{\frac{U_i}{V_{A_i}} = 2^{\frac{1}{4}} S^{-\frac{1}{2}}}$

a. S is a very large number for magnetospheric plasmas due to low resistivity.

b. Thus, $\boxed{\text{Inflow rate is } \underline{\text{very}} \text{ Slow.}}$

c. This does not agree with rapid reconnection rates inferred from observations.

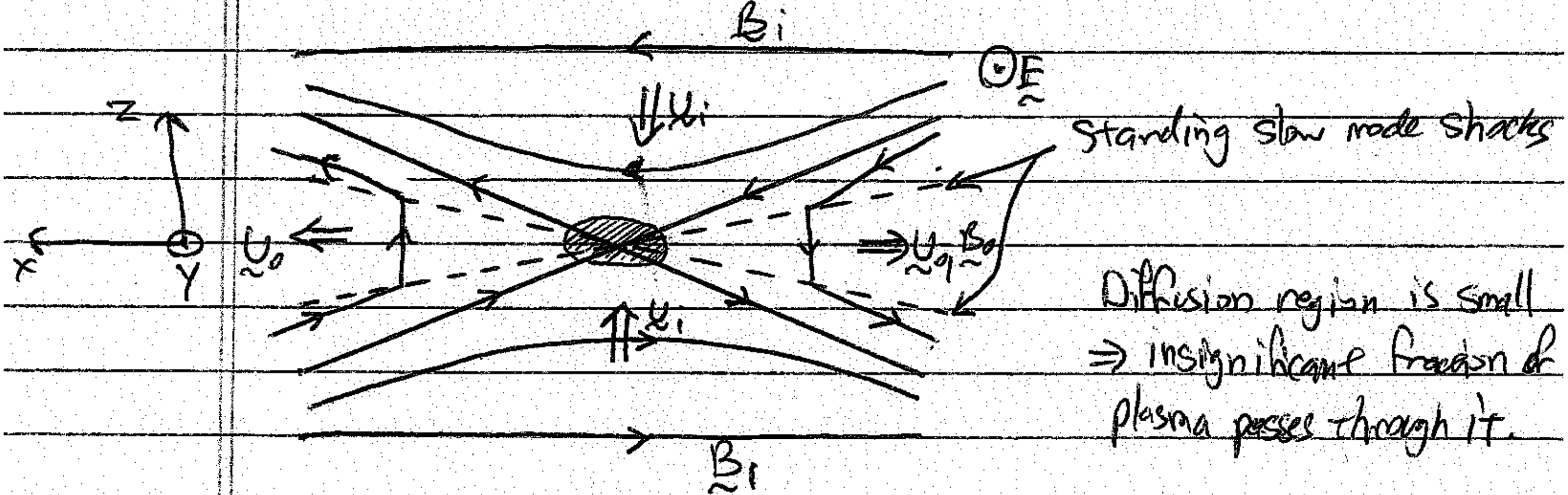
Lecture #14 (continued) Howes ⑦

III. Petschek Reconnection

A. Reconnection at X-point rather than Current Sheet

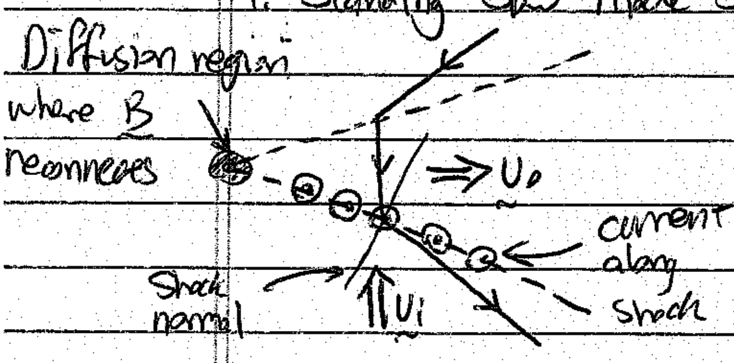
1. Proposed that not all plasma needs to flow through diffusion region ⇒ Alleviates bottleneck and allows faster rate
2. Acceleration can occur outside of diffusion region at a pair of standing slow shocks.

3. Petschek Reconnection:



Diffusion region is small ⇒ insignificant fraction of plasma passes through it.

4. Standing Slow mode Shocks



a) Magnetic field bends through shocks toward normal ⇒ Slow mode shock (See Lec #9)

b) Slow mode shocks are also current sheets

c) $\underset{\sim}{j} \times \underset{\sim}{B}$ force due to current sheet accelerates the flow

d) Plasma is also compressed at shock, $\rho_0 > \rho_i$.

5. Detailed analysis yields $\boxed{\frac{U_i}{V_{Ai}} \leq 0.1}$, much larger than $S^{-\frac{1}{2}}$

⇒ However, there remain problems with the Petschek model.