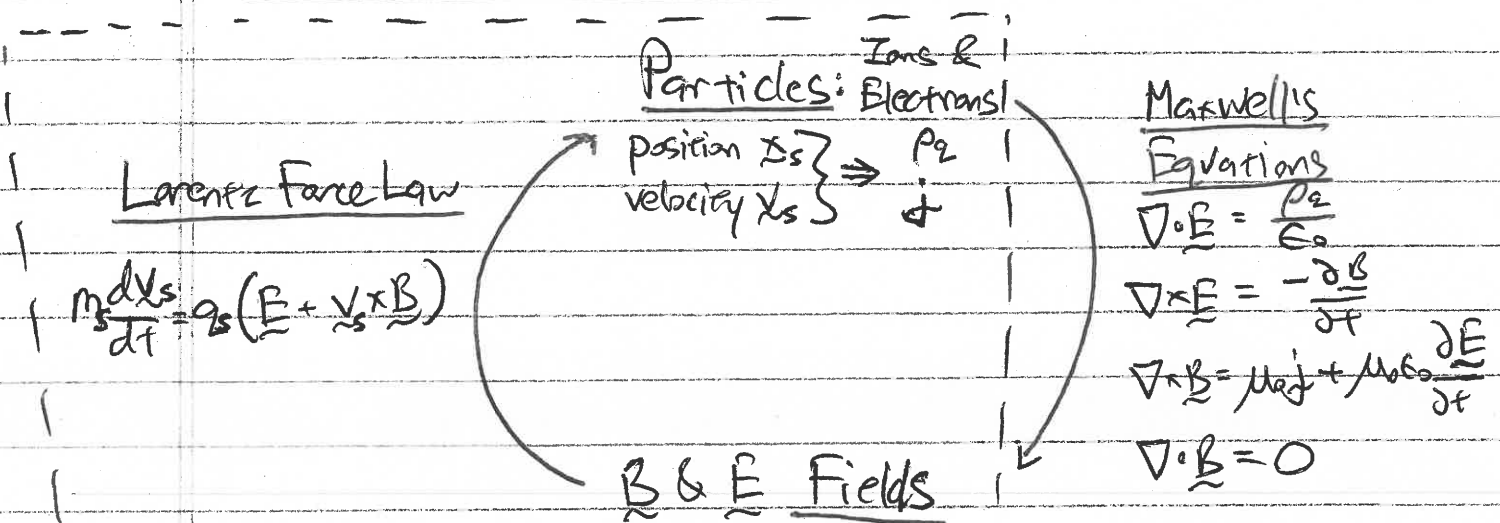


## Lecture #2 - Single Particle Motion

### I. Overall Framework of Plasma Physics



### Single Particle Motion Description

What we want to study is how charged particles move in prescribed  $\mathbf{E}$  &  $\mathbf{B}$  fields.

### II. Larmor Motion: Constant, Uniform $\mathbf{B}$ with $\mathbf{E} = 0$

A. 1. Non relativistic limit  $v \ll c$

2. Drop subscript "s" for species

3. Thus, for  $\mathbf{E} = 0$ ,  $m \frac{d\mathbf{v}}{dt} = q \mathbf{v} \times \mathbf{B}$

4. Take  $\mathbf{B} = B_0 \hat{\mathbf{z}}$  where  $B_0 = \text{const}$

#### B. Solution:

1.  $\frac{dv_x}{dt} = \frac{qB_0}{m} v_y$  ①

$\frac{dv_y}{dt} = -\frac{qB_0}{m} v_x$  ②

$\frac{dv_z}{dt} = 0 \Rightarrow v_z = \text{constant}$

## II. $B_0$ (Continued)

Pages ③

2. Define: Cyclotron Frequency:  $\Omega \equiv \frac{qB_0}{m}$

3. To solve: a. Take  $\frac{d}{dt}$  ① and substitute ③

$$\frac{d^2 v_x}{dt^2} = -\Omega^2 v_x$$

b. General Solution:  $v_x = A e^{-i\Omega t} + B e^{i\Omega t}$

c. Apply Initial Conditions to solve for A & B

i. Take  $v_x = v_\perp$ ,  $v_y = 0$  at  $t=0$ ,  $\Rightarrow A = B = \frac{v_\perp}{2}$

ii. Let  $v_z = v_{||}$  at  $t=0$  also.

d. Thus,  $v_x = v_\perp \cos \Omega t$

$$v_y = -v_\perp \sin \Omega t$$

$$v_z = v_{||}$$

e. Solve for position:  $\frac{dx}{dt} = v \Rightarrow$

$$x = \frac{v_\perp}{\Omega} \sin \Omega t + x_0$$

$$y = \frac{v_\perp}{\Omega} \cos \Omega t + y_0$$

$$z = v_{||} t + z_0$$

4. Define: Larmor Radius  $r_L \equiv \frac{v_\perp}{\Omega} = \frac{mv_\perp}{qB_0}$

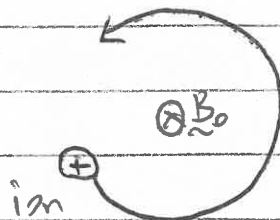
5. Summary:

a.  $\underline{x}(t) = r_L (\sin \Omega t \hat{x} + \cos \Omega t \hat{y}) + v_{||} t \hat{z} + \underline{x}_0$

b.  $\underline{v}(t) = v_\perp (\cos \Omega t \hat{x} - \sin \Omega t \hat{y}) + v_{||} \hat{z}$

## C. Properties

1. Diamagnetic:



Field due to Larmor motion opposes mean field

## II.C. (Continued)

Hawes ③

### 2. Constant Energy:

$$a. \frac{dE}{dt} = \frac{d}{dt} \left( \frac{1}{2} m v^2 \right) = \underline{v} \cdot \left( m \frac{d\underline{v}}{dt} \right) = \underline{v} \cdot [q_s (\underline{v} \times \underline{B})] = 0$$

b. Thus,  $v_{\perp} = \text{constant}$ .

## III. $\underline{E} \times \underline{B}$ Drift: Constant, Uniform $\underline{B}$ and $\underline{E}$

### A. Drift Motion $\underline{B} = B_0 \hat{z}$

$$1. m \frac{d\underline{v}}{dt} = q (\underline{E} + \underline{v} \times \underline{B})$$

2. What velocity  $\underline{v}$  leads to RHS = 0?  $\Rightarrow$  no acceleration  $\Rightarrow$  drift

$$a. \underline{E} = -\underline{v} \times \underline{B}$$

$$b. \text{Cross with } \underline{B}: \underline{E} \times \underline{B} = -(\underline{v} \times \underline{B}) \times \underline{B} = B_0^2 (\underline{v} - v_z \hat{z})$$

$$c. \text{Thus, } \underbrace{\underline{v} - v_z \hat{z}}_{\text{Perpendicular to } \underline{B}_0} = \frac{\underline{E} \times \underline{B}}{B_0^2}$$

$$3. \text{Define "E cross B" velocity } \underline{v}_E \equiv \frac{\underline{E} \times \underline{B}}{B_0^2}$$

### B. Motion in $\underline{E} \times \underline{B}$ drift Frame

$$1. \text{Solve for velocity } \underline{v} \text{ in } \underline{E} \times \underline{B} \text{ frame: } \underline{v} = \underline{u} + \underline{v}_E$$

$$2. \text{Substitute for } \underline{v}: m \frac{d\underline{u}}{dt} + m \frac{d\underline{v}_E}{dt} = q (\underline{E} + \underline{v}_E \times \underline{B} + \underline{u} \times \underline{B})$$

$$a. \underline{v}_E \times \underline{B} = \frac{(\underline{E} \times \hat{z}) \times \hat{z} B_0^2}{B_0^2} = E_z \hat{z} - \underline{E}$$

$$b. \text{Thus } m \frac{d\underline{u}}{dt} = q (E_z \hat{z} + \underline{u} \times \underline{B})$$

$$3. \text{Parallel Motion } (\hat{z}): m \frac{du_z}{dt} = q E_z \Rightarrow \underline{u}_z = \frac{q E_z}{m} t + u_{z0}$$

### III. B. (Continued)

Hines (4)

4. Perpendicular Motion:  $\underline{v}_\perp = v - v_z \hat{z}$

a.  $m \frac{d\underline{v}_\perp}{dt} = q(\underline{v}_\perp \times \underline{B})$  This is identical to the case with  $\underline{E} = 0$ .

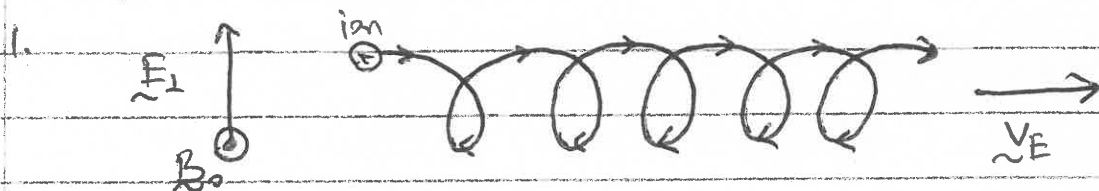
b. Thus,  $\underline{v}_\perp = v_\perp (\cos \Omega t \hat{x} - \sin \Omega t \hat{y})$

c. In the  $\underline{E} \times \underline{B}$  drift frame, you have the usual Larmor motion

5. Full Solution:

$$\underline{v} = \underbrace{\left( \frac{q E_z}{m} + v_{z0} \right) \hat{z}}_{\text{Parallel Motion}} + \underbrace{v_\perp (\cos \Omega t \hat{x} - \sin \Omega t \hat{y})}_{\text{Larmor Motion}} + \underbrace{\left( \frac{\underline{E} \times \underline{B}}{B_0^2} \right)}_{\underline{E} \times \underline{B} \text{ drift}}$$

### C. Physical Picture:



a. Acceleration by  $\underline{E} \Rightarrow r_L$  increases } This asymmetry leads  
 Deceleration by  $\underline{E} \Rightarrow r_L$  decreases } to the drift

2.  $\underline{E} \times \underline{B}$  drift is independent of charge.  $\Rightarrow$  No net current due to  $\underline{E} \times \underline{B}$  drift.



### IV Multiple Timescale Methods

A.1. A powerful approach to solving many plasma physics problems is the use of multiple timescale methods.

2. In many problems, different components of the motion occur on disparate timescales.

#### IV. A. 2. (Continued)

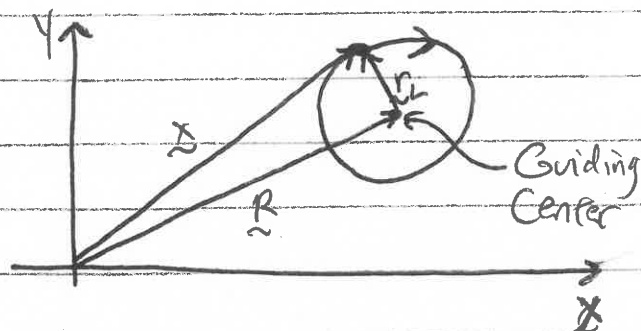
Howes (5)

a. For Example,  $\underline{E} \times \underline{B}$  drift

Decomposition  
of Motion:

- i. Rapid Larmor motion about field line
- ii. Slow drift across field line.

3. Define: Guiding Center



a. Position can be split into  
Guiding Center  $\underline{R}$   
plus Larmor motion  $\underline{r}_L$

$$\underline{x} = \underline{R} + \underline{r}_L$$

4. Basic concept for multiscale methods:

a. Average over fast timescale motion:

$$\int_0^{2\pi} dt \underline{r}_L(t) = 0$$

b. This leaves the slow timescale drift motion  $\underline{R}(t)$ .

#### V. $\nabla B$ & Curvature Drifts: Constant, Non-uniform $\underline{B}$ fields

A. In the fusion program, magnetic fields used to confine the plasma are neither straight nor uniform. We want to understand particle motion in  $\underline{B}$  fields of varying strength and curved  $\underline{B}$  fields.

#### B. Drift due to a General Force $\underline{F}$

1. Analogous to  $\underline{E} \times \underline{B}$  drift, take  $m \frac{d\underline{v}}{dt} = q \underline{v} \times \underline{B} + \underline{F}$

$$\Rightarrow \underline{v}_D = \frac{1}{q} \frac{\underline{F} \times \underline{B}}{B_0^2}$$

2. NOTE: The direction of this drift depends on the charge sign.

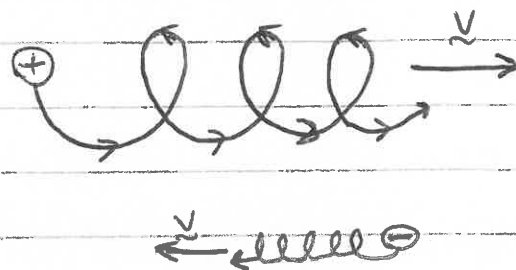
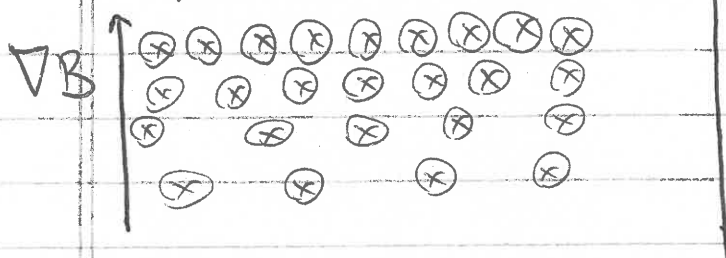
## IV. (Continued)

Hawes ⑥

### C. The $\nabla B$ ("GradB") Drift

1. Simplest Case  $\nabla B \perp \underline{B}$

2. Physical Picture:



a. Stronger  $B \Rightarrow$  smaller  $r_L$   
 Weaker  $B \Rightarrow$  larger  $r_L$

3. Multiscale Approach:  
 a. Small scale: Larmor Radius  $r_L = \frac{v_{\perp}}{\Omega}$   
 b. Large scale:  $B$  Scale length  $L \equiv \left(\frac{\nabla B}{B}\right)^{-1}$

c. We may use a perturbative approach in the small expansion parameter  $\epsilon \equiv \frac{r_L}{L} \ll 1$

d. We may derive the average force on the particle  $\langle \underline{F} \rangle$  (averaged over the Larmor period  $T = \frac{2\pi}{\omega_c}$ ) due to the  $\nabla B$ .

$$\langle \underline{F} \rangle = -\frac{q}{2} \frac{v_{\perp}^2}{\Omega} \nabla B$$

4. The result is the  $\nabla B$  drift,

$$\underline{v}_{\nabla B} = -\frac{v_{\perp}^2}{2\Omega} \frac{\nabla B \times \underline{B}}{B^2}$$

a. NOTE: Since  $\Omega = \frac{qB}{m}$ , the  $\nabla B$  drift depends on charge.  
 $\Rightarrow$  Ions and electrons drift in opposite directions

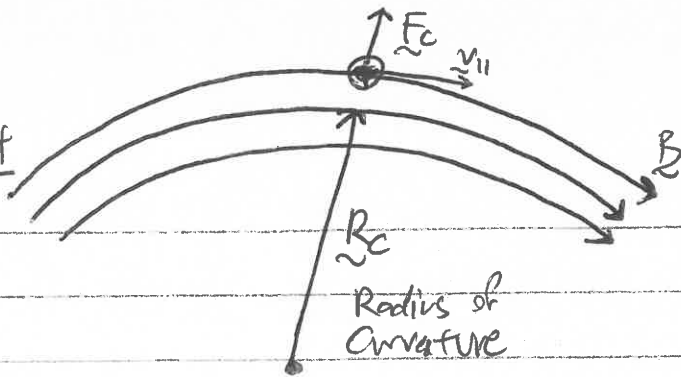
b. Drift magnitude depends on perpendicular energy  $\frac{1}{2} m v_{\perp}^2$ .

# V. (Continued)

Howes 7

## D. Curvature Drift

### 1. Physical Picture



### 2. Simple Estimate:

a. For a particle moving along a circular path along  $\underline{B}$ , the centrifugal force felt by the particle is

$$\underline{F}_c = \frac{m v_{||}^2}{R_c} \hat{r} = \frac{m v_{||}^2}{R_c} \underline{\hat{r}}$$

b. Treating this as the general force  $\underline{F}$ , we find

Curvature Drift 
$$\underline{v}_c \equiv \frac{m v_{||}^2}{q B^2} \frac{\underline{R}_c \times \underline{B}}{R_c^2} = \frac{v_{||}^2}{\Omega B} \frac{\underline{R}_c \times \underline{B}}{R_c^2}$$

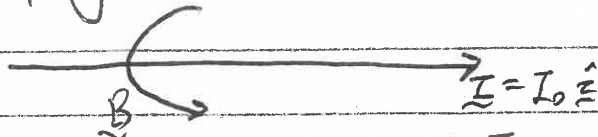
3. Properties: a. Depends on parallel energy  $\frac{1}{2} m v_{||}^2$

b. Again, ions & electrons drift in opposite directions

4. NOTE: When  $\underline{B}$  field lines are curved, there is typically also a gradient in  $|\underline{B}|$ , so both  $\nabla B$  & curvature drifts will be important.

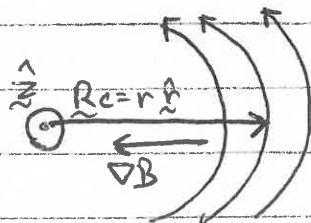
## E. Example: Current Carrying Wire

Consider a wire carrying a current  $\underline{I} = I_0 \hat{z}$



1. In cylindrical coordinates  $(r, \phi, z)$ ,  $\underline{B} = \frac{\mu_0 I_0}{2\pi r} \hat{\phi}$

2. End on view



From NRL Plasma Formulary p.6,

$$\nabla B = \frac{\partial B}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial B}{\partial \phi} \hat{\phi} + \frac{\partial B}{\partial z} \hat{z} = -\frac{\mu_0 I_0}{2\pi r^2} \hat{r}$$

## V. E. (Continued)

Hawes (8)

### 3. $\nabla B$ Drift:

$$\underline{v}_{\nabla B} = -\frac{v_{\perp}^2}{2\Omega} \frac{\nabla B \times \underline{B}}{B_0^2} = -\frac{v_{\perp}^2}{2\Omega} \frac{\left(-\frac{\mu_0 I_0}{2\pi r^2} \hat{r}\right) \times \left(\frac{\mu_0 I_0}{2\pi r} \hat{\phi}\right)}{\left(\frac{\mu_0 I_0}{2\pi r}\right)^2} = +\frac{v_{\perp}^2}{2\Omega r} \hat{z}$$

### 4. Curvature Drift:

$$\underline{v}_c = \frac{v_{\parallel}^2}{\Omega B} \frac{R_c \times \underline{B}}{R_c^2} = \frac{v_{\parallel}^2}{\Omega \left(\frac{\mu_0 I_0}{2\pi r}\right)} \frac{(r \hat{r}) \times \left(\frac{\mu_0 I_0}{2\pi r} \hat{\phi}\right)}{r^2} = \frac{v_{\parallel}^2}{\Omega r} \hat{z}$$

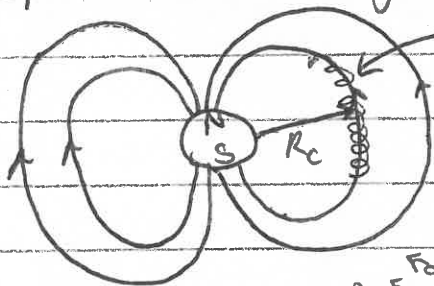
### 5. Net Drift: $\underline{v} = \underline{v}_{\nabla B} + \underline{v}_c = \frac{1}{\Omega r} \left(\frac{v_{\perp}^2}{2} + v_{\parallel}^2\right) \hat{z}$

a. NOTE:  $\frac{1}{\Omega r} = \frac{m}{qBr} = \frac{m}{q\mu_0 I_0 r}$ , so  $\underline{v} = \frac{2\pi}{q\mu_0 I_0} \left(\frac{mv_{\perp}^2}{2} + mv_{\parallel}^2\right) \hat{z}$

Velocity is independent of  $r$ !

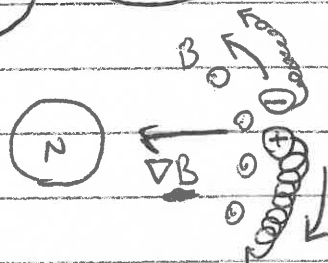
## F. Example: Earth's Magnetosphere

Side View



1. Particles trapped in Earth's dipole field experience  $\nabla B$  & curvature drifts

Top View



2. These drifts produce the "ring current" in the westward direction

3. Strength of ring current is proportional to the energy of the particles.  
 $\Rightarrow$  Magnetic Storms!