Lecture #23: Magnetic Buoyancy and the Parker Instability

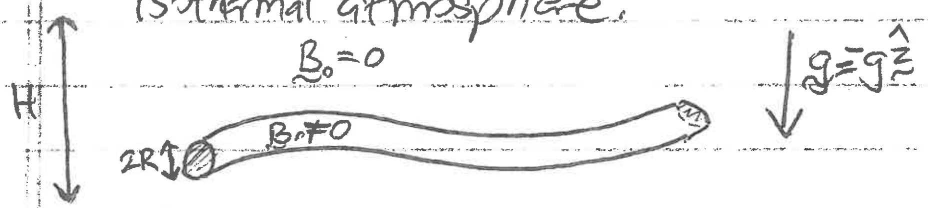
I. Magnetic Buoyancy:

A. Importance

1. In compressible fluids, magnetic buoyancy serves to expel magnetic flux from the Sun, stars, accretion disks, and galaxies.
2. Magnetic buoyancy is the driver of magnetic activity in astronomical objects. For example, the solar dynamo is believed to generate active regions, and lead to energy transport into the corona, which ultimately accelerates the solar wind.

B. Why are magnetic fields buoyant?

1. Consider an isolated magnetic flux tube in a plane isothermal atmosphere.

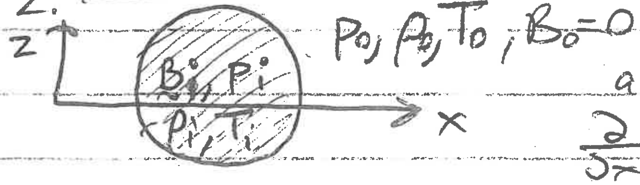


a. Outside the magnetic flux tube $B_0 = 0$, so we have an isothermal plane atmosphere (see Lec # 19, II B.9.c.)

$$p_0(z) = p_0(0)\, e^{-\frac{z}{H}} \quad \text{where} \quad H = \frac{2T}{m_p g} \text{ is scale height.}$$

b. For a flux tube radius $R \ll H$, conditions around the flux tube are almost uniform, p_0 & ρ_0, where $p_0 = \frac{2T}{m_p}\rho_0 = C_s^2 \rho_0$ in isothermal limit.

2. Cross-Section



a. In x-direction, pressure balance is

$$\frac{\partial}{\partial x}\left(p_i + \frac{B_i^2}{8\pi}\right) = \frac{\partial}{\partial x}(p_0)$$

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I. B. 2. (Continued)

b. Integrating, horizontal pressure balance yields $p_i + \frac{B_i^2}{8\pi} = p_o$

3. Since $T_i = T_o = T$ and $p = \frac{2T}{m_p}\rho$,

a. $\rho_i \frac{2T}{m_p} = \rho_o \frac{2T}{m_p} - \frac{B_i^2}{8\pi} \Rightarrow \rho_i = \rho_o - \frac{B_i^2 m_p}{16\pi T} < \rho_o$

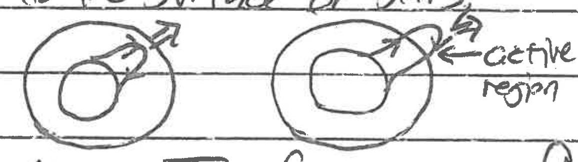
b. ~~Thus, since~~ $\rho_i < \rho_o$, the flux tube is buoyant.

c. Vertical Force:
Gravity: $-\rho_i g \hat{\underset{\sim}{z}} < -\rho_o g \hat{\underset{\sim}{z}} \rightarrow$ So flux tube has acceleration upward relative to surrounding plasma.

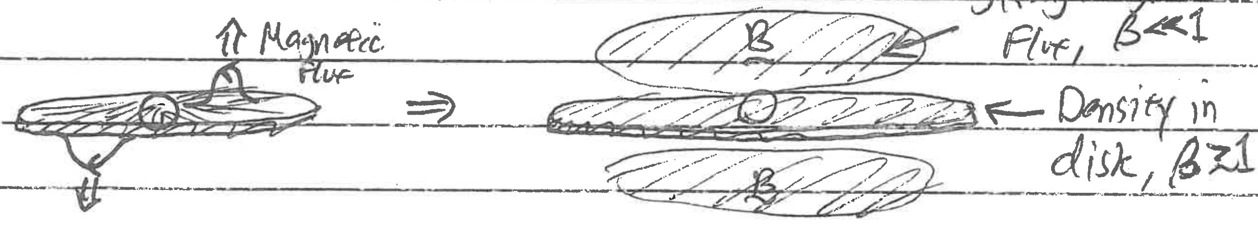
d. $\Delta \rho g = -\rho_i g + \rho_o g = \cancel{-\rho_o g} + \frac{B_i^2}{8\pi}\frac{m_p g}{2T} + \cancel{\rho_o g} = \frac{B_i^2}{8\pi H}$ ← difference in gravitational accel.

C. Effects of Magnetic Buoyancy in Astrophysical Objects

1. Stars: Drives magnetic flux to the surface of stars, driving magnetic activity.



2. Galactic disks and accretion disks: Transfers magnetic flux to the halo or corona of disks



II. Magnetic Buoyancy Instabilities:

A. Equilibrium

1. As shown above, a horizontal, isothermal isolated flux tube cannot be in equilibrium.

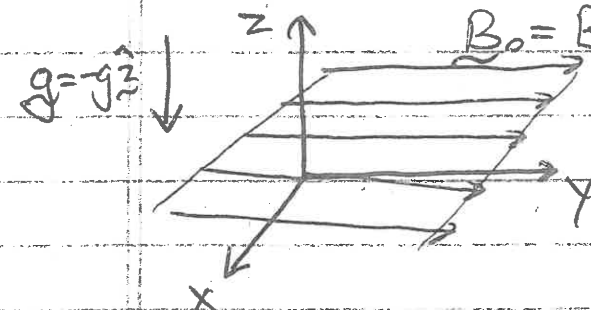
2. But, a horizontal, isothermal magnetic flux sheet can be in equilibrium (density in horizontal direction is constant).

II. A. (Continued)

3. However, a flux sheet is susceptible to magnetic buoyancy instabilities.

B. Types of Magnetic Buoyancy Instabilities

1. Consider the case of a 2-D flux sheet.

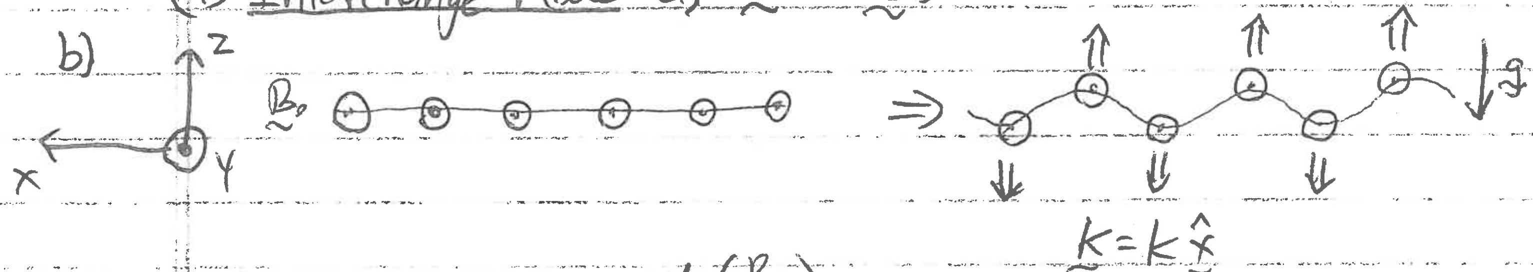


a. $\underline{B}_0$ in $\hat{y}$-direction

b. $\underline{g}$ravity in $-\hat{z}$-direction

2. Two Types of Instabilities:

(1) Interchange Mode a) $\underline{k} \perp \underline{B}_0$



c) Instability Condition: $\frac{d}{dz}\left(\frac{B_0}{\rho_0}\right) < 0$

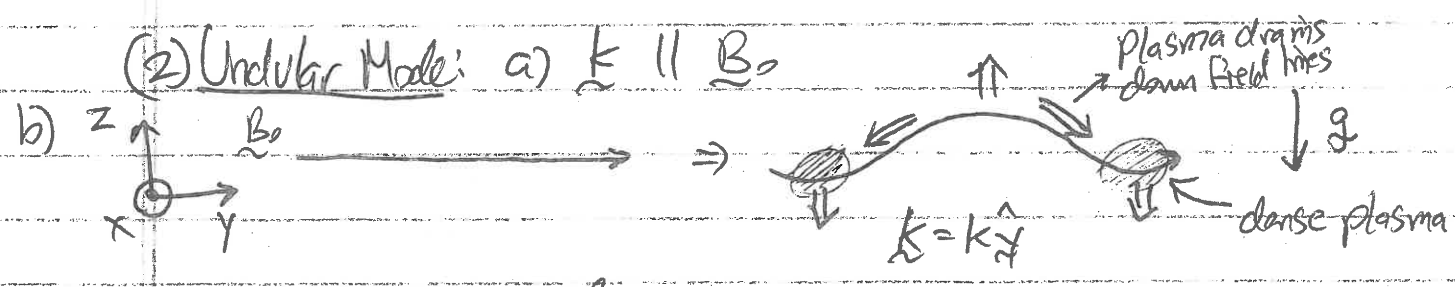
d) Unstable Wavenumbers: Any $\underline{k}$, with faster growth at large k

e) Also called: i) Magnetic Rayleigh-Taylor Instability

ii) Flute Instability (lab plasmas)

iii) Kruskal-Schwarzschild Instability

(2) Undular Mode: a) $\underline{k} \parallel \underline{B}_0$



c) Instability Condition: $\frac{d}{dz} B_0 < 0$

II. B2 (Continued)

(2) (Continued)

d) Unstable Wavenumbers: $kH \lesssim 2$ for isothermal atmosphere

Small scale (high k) modes are stabilized by magnetic tension.

e) Also called: i) Parker Instability (astrophysics)

ii) Ballooning Instability (fusion) (driven by pressure gradients)

III. The Parker Instability

Ref: Parker, E.N., The Dynamical State of the Interstellar Gas and Field

ApJ, 145, 811 (1966).

A. MHD Equations:

1. Continuity: $\frac{\partial \rho}{\partial t} + \underset{\sim}{U}\cdot\nabla\rho = -\rho\nabla\cdot\underset{\sim}{U}$

2. Momentum: $\rho\frac{\partial U}{\partial t} + \rho\underset{\sim}{U}\cdot\nabla\underset{\sim}{U} = -\nabla\left(p+\frac{B^2}{8\pi}\right) + \frac{\underset{\sim}{B}\cdot\nabla\underset{\sim}{B}}{4\pi} \underbrace{-\rho g(z)\hat{\underset{\sim}{z}}}$

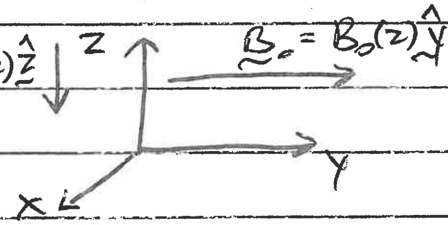
3. Induction: $\frac{\partial \underset{\sim}{B}}{\partial t} + \underset{\sim}{U}\cdot\nabla\underset{\sim}{B} = -\underset{\sim}{B}\,\nabla\cdot\underset{\sim}{U} + \underset{\sim}{B}\cdot\nabla\underset{\sim}{U}$

Gravitational Acceleration $\underset{\sim}{g} = -g(z)\hat{\underset{\sim}{z}}$

4. Eq. of State: $\frac{\partial p}{\partial t} + \underset{\sim}{U}\cdot\nabla p = -\gamma p\nabla\cdot\underset{\sim}{U}$

B. Equilibrium: $\underset{\sim}{g} = -g(z)\hat{\underset{\sim}{z}}$

1. Consider a system:



Vertical gravity

horizontal $\underset{\sim}{B}$ field

2. Vertical Force Balance

a) Equilibrium: $\underset{\sim}{U}_0 = 0$, Steady-state $\frac{\partial}{\partial t} = 0$

b) z-component of momentum eq:

$$-\frac{\partial}{\partial z}\left(p_0 + \frac{B_0^2}{8\pi}\right) - \rho_0 g(z) = 0$$

c) General Equilibrium $\boxed{\frac{\partial}{\partial z}\left(p_0(z) + \frac{B_0^2(z)}{8\pi}\right) = -\rho_0(z)\,g(z)}$

III. B. (Continued)

3. Simplifying Assumptions for Equilibrium:

a. Plasma Beta, $\beta = \frac{8\pi p_0}{B_0^2}$ is independent of z

b. Isothermal equilibrium ($\gamma=1$), $p_0 = c_s^2 \rho_0$ where c_s^2 = constant

c. Constant Gravitational Acceleration, $\mathbf{g} = -g\hat{\mathbf{z}}$

4. a. $$\frac{\partial}{\partial z}\left[p_0\left(1+\frac{1}{\beta}\right)\right] = \left(1+\frac{1}{\beta}\right)\frac{\partial p_0}{\partial z} = -\rho_0 g$$

b. $$\frac{\partial p_0}{\partial z} = c_s^2 \frac{\partial \rho_0}{\partial z} \Rightarrow \left(1+\frac{1}{\beta}\right)c_s^2 \frac{\partial \rho_0}{\partial z} = -\rho_0 g$$

c. $$\left(1+\frac{1}{\beta}\right)c_s^2 \int_0^z \frac{\partial \rho_0}{\rho_0} = \int_0^z -g\,\partial z \Rightarrow \rho_0(z) = \rho_0(0)\, e^{-\frac{g\beta}{c_s^2(1+\beta)}z}$$

5. Define: Scale Height: $\boxed{H \equiv \frac{c_s^2(1+\beta)}{g\beta}}$

Isothermal Atmosphere: $\boxed{\rho_0(z) = \rho_0(0)\, e^{-\frac{z}{H}}}$

6. Note: a. $\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z} = \frac{-1}{H}$ b. $\frac{1}{p_0}\frac{\partial p_0}{\partial z} = \frac{-1}{H}$ c. $\frac{1}{B_0}\frac{\partial B_0}{\partial z} = \frac{-1}{2H}$

d. Therefore, this isothermal atmosphere is characterized by the scale height H (function of g, β, T)

III. (Continued)

C. Linear Dispersion Relation for Parker Instability

1. Setup and Assumptions:

a. Horizontal Magnetic Field: $\underset{\sim}{B}_0 = B_0(z)\hat{y}$

b. Constant Vertical Gravity: $g = -g\hat{z}$

c. Resting, Steady State Equilibrium, $\underset{\sim}{U}_0 = 0$

d. Isothermal Equilibrium with scale height $H \equiv \frac{c_s^2(1+\beta)}{g\,\beta}$

e. Perturbations have arbitrary γ: $\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0 \Rightarrow$ Eq. of State

f. Constant Plasma Beta, β, independent of z

g. Allow only $\frac{\partial}{\partial y} \neq 0$ (undular mode, parallel perturbations)

$\frac{\partial}{\partial x} = 0$ (no interchange mode), $\frac{\partial}{\partial z} = 0$ for simplicity

2. It can be shown that for $\frac{\partial}{\partial x} = 0$ and $\frac{\partial}{\partial z} = 0$, there is no instability drive for U_x and B_x. Thus, we take $U_x = 0$, $B_x = 0$. (Solve for $U_y, U_z, B_y, B_z, \rho_1, p_1$)

a. Notation: $\underset{\sim}{U}_1 = U_x\hat{x} + U_y\hat{y} + U_z\hat{z}$

$\underset{\sim}{B}_1 = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$

3. Usual Steps to Solve for Linear Dispersion Relation

(1) Linearize Equations

(2) Fourier Transform $e^{i(ky-\omega t)}$

(3) Solve $\underset{\approx}{D} \cdot \underset{\sim}{U}_1 = 0$, $\Rightarrow |\underset{\approx}{D}| = 0$

D. Linearization:

1. $\underset{\sim}{B} = B_0(z)\hat{y} + \epsilon \underset{\sim}{B}_1$

$\underset{\sim}{U} = \epsilon \underset{\sim}{U}_1$

$p = p_0(z) + \epsilon p_1$

$\rho = \rho_0(z) + \epsilon \rho_1$

III. D. (Continued)

2. Equilibrium Gradients:

a. Gradients of B_0, p_0, & ρ_0 in z will lead to new terms in Linearized equations $\Rightarrow$ These terms can drive instability!

b. When Fourier transforming linearized equations, you do NOT Fourier transform equilibrium gradients. Only linear perturbations $\rho_1, \underset{\sim}{U}_1, \underset{\sim}{B}_1$, & p_1 have the form $\sim e^{i(ky-\omega t)}$

c. Notation: $B_0' = \frac{\partial B_0}{\partial z}$, $p_0' = \frac{\partial p_0}{\partial z}$, $\rho_0' = \frac{\partial \rho_0}{\partial z}$

3.
$$\frac{\partial \rho_1}{\partial t} + U_z \rho_0' = -\rho_0 \nabla \cdot \underset{\sim}{U}_1$$

$$\frac{\partial \underset{\sim}{U}_1}{\partial t} = -\nabla \frac{p_1}{\rho_0} - V_A^2 \frac{B_0'}{B_0}\frac{B_y}{B_0}\hat{\underset{\sim}{z}} - V_A^2 \nabla \frac{B_y}{B_0} + V_A^2 \frac{\partial}{\partial y}\left(\frac{\underset{\sim}{B}_1}{B_0}\right) + V_A^2 \frac{B_0'}{B_0}\frac{B_z}{B_0}\hat{\underset{\sim}{y}} - \frac{\rho_1}{\rho_0} g \hat{\underset{\sim}{z}}$$

$$\frac{\partial \underset{\sim}{B}_1}{\partial t} + U_z B_0' \hat{\underset{\sim}{y}} = -B_0 \hat{\underset{\sim}{y}}\, \nabla \cdot \underset{\sim}{U}_1 + B_0 \frac{\partial \underset{\sim}{U}_1}{\partial y}$$

$$\frac{\partial p_1}{\partial t} + U_z p_0' = -\gamma p_0 \nabla \cdot \underset{\sim}{U}_1$$

4a. Simplifying using $\frac{\partial}{\partial x} = 0$, $\frac{\partial}{\partial z} = 0$, $U_x = 0$, & $B_x = 0$:

$$\frac{\partial \rho_1}{\partial t} = \underset{\sim\sim}{-U_z \rho_0'} - \rho_0 \frac{\partial U_y}{\partial y}$$

$$\frac{\partial U_y}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p_1}{\partial y} + \underset{\sim\sim}{V_A^2 \frac{B_0'}{B_0}\frac{B_z}{B_0}}$$

$$\frac{\partial U_z}{\partial t} = \underset{\sim\sim}{-V_A^2 \frac{B_0'}{B_0}\frac{B_y}{B_0}} + V_A^2 \frac{\partial}{\partial y}\left(\frac{B_z}{B_0}\right) - \frac{\rho_1}{\rho_0} g$$

$$\frac{\partial B_y}{\partial t} = \underset{\sim\sim}{-U_z B_0'}$$

$$\frac{\partial B_z}{\partial t} = B_0 \frac{\partial U_z}{\partial y}$$

$$\frac{\partial p_1}{\partial t} = \underset{\sim\sim}{-U_z p_0'} - \gamma p_0 \frac{\partial U_y}{\partial y}$$

III. E. Remaining Steps to Linear Dispersion Relation:

1. Fourier Transform
2. a. Substitute ρ_1 & B_z into eq. for U_y
   b. Substitute B_y, B_z, & ρ_1 into eq. for U_z

$\Rightarrow$ Obtain coupled algebraic equations for U_y & U_z:

$$\underset{\approx}{D} \cdot \underset{\sim}{U}_1 = \begin{pmatrix} D_{yy} & D_{yz} \\ D_{zy} & D_{zz} \end{pmatrix} \begin{pmatrix} U_y \\ U_z \end{pmatrix} = 0$$

3. Dispersion Relation $|\underset{\approx}{D}| = D_{yy} D_{zz} - D_{yz} D_{zy} = 0$

4. Apply isothermal equilibrium, $\frac{p_0'}{p_0} = \frac{\rho_0'}{\rho_0} = \frac{2B_0'}{B_0} = -\frac{1}{H}$ to simplify.

5. Ultimately, one obtains (Parker (1966) Eq. (III-14) with $\frac{\partial}{\partial \xi}=0$, etc.)

$$\boxed{\bar{\omega}^4 + \bar{\omega}^2\left[-\left(\frac{2}{\beta}+\gamma\right)\left(1+\frac{1}{4k^2H^2}\right)\right] + \left[\frac{2\gamma}{\beta} - \frac{\left(1+\frac{1}{\beta}-\gamma\right)\left(1+\frac{1}{\beta}\right)-\frac{\gamma}{2\beta}}{k^2H^2}\right] = 0}$$

where $\bar{\omega} = \frac{\omega}{kc_s}$ is dimensionless frequency.

a. NOTE: $\bar{\omega} = \bar{\omega}(\gamma, \beta, kH)$

F. Properties of Parker Instability:

1. Equation has form $\bar{\omega}^4 - b\bar{\omega}^2 + c = 0$ (quartic)

a. Thus $\bar{\omega}^2 = \frac{b}{2} \pm \frac{1}{2}\sqrt{b^2-4c}$

where $b = \left(\frac{2}{\beta}+\gamma\right)\left(1+\frac{1}{4k^2H^2}\right)$ and $c = \frac{2\gamma}{\beta} - \frac{\left(1+\frac{1}{\beta}-\gamma\right)\left(1+\frac{1}{\beta}\right)-\frac{\gamma}{2\beta}}{k^2H^2}$

2. Stability only if $\bar{\omega}^2 \geq 0$ and $\bar{\omega}^2$ real

This requires a) $b^2 - 4c > 0 \rightarrow \bar{\omega}^2$ real

b) $b > \sqrt{b^2-4c} \Rightarrow c \geq 0 \Rightarrow \bar{\omega}^2 \geq 0$

III. F. (Continued)

3. Thus, the plasma is unstable if $\boxed{c < 0}$

4. Therefore, instability occurs when

a. $$\frac{2\gamma}{\beta} < \frac{\left(1+\frac{1}{\beta}-\gamma\right)\left(1+\frac{1}{\beta}\right)-\frac{\gamma}{2\beta}}{k^2H^2}$$

b. $$\boxed{(\beta+1-\gamma\beta)(\beta+1)-\frac{\gamma\beta}{2} > 2\gamma\beta k^2H^2}$$

c. For any possible k, the least restrictive condition occurs for $kH \rightarrow 0$ (large wavelengths are most unstable).

$$(\beta+1-\gamma\beta)(\beta+1)-\frac{\gamma\beta}{2} > 0$$

d. For the isothermal case $\gamma=1$, this yields $\frac{\beta}{2}+1 > 0 \Rightarrow$ Always true!

Thus, an isothermal atmosphere is always unstable to the Parker Instability

5. In general, this instability condition can be written,

$$\boxed{\gamma - 1 < \frac{\frac{1}{2\beta}+\frac{1}{\beta^2}}{\left(1+\frac{3}{2\beta}\right)}}$$

a. In the unmagnetized limit $\beta \rightarrow \infty$, $\Rightarrow \gamma < 1$ is unstable.

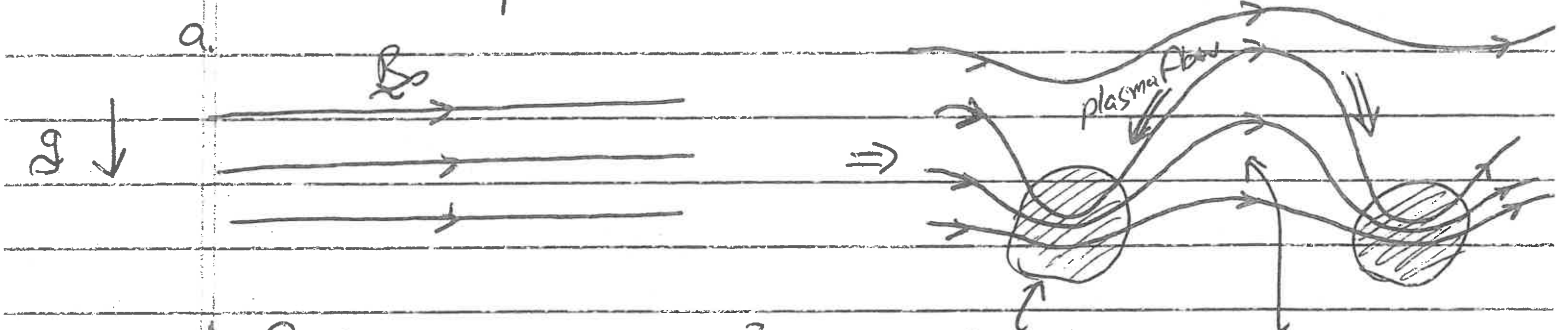
So, the isothermal atmosphere is stable for thermal gas alone!

b. The magnetic field makes the system unstable.
Must have $\gamma > 1$ to maintain stability.

III. G. Consequences of the Parker Instability:

1. Interstellar plasma tends to collect in discrete clouds

a.



b. Clouds separations are $10-10^3$ pc.

$H \sim 100$ pc in galaxy, so

$kH \sim 1 \Rightarrow \frac{2\pi}{2L} H \sim 1 \rightarrow L \sim \pi H \sim 300$ pc

2. Cosmic rays can also contribute to the buoyancy instability, leading to thermal gas draining down field lines.

3. a. For typical interstellar medium densities $n \lesssim 10$ cm^{-3}, Magnetic field in ISM cannot exceed $B \gtrsim 5 \mu G$.

b. A stronger B would be unstable and rise out of the galactic disk.

c. Parker instability gives limit on density given magnetic field strength, or limits on magnetic field strengths given the density.

4. Magnetic tension stabilizes the instability, so there is typically a maximum unstable wavenumber (minimum unstable wavelength) for a particular choice of γ, β. [Remember, $\bar{\omega} = \bar{\omega}(\gamma, \beta, kH)$].