Lecture #5: MHD, Dimensionless Numbers, Frozen-In Flux, and Magnetic Diffusion

I. Magnetohydrodynamics (MHD)

1. MHD is a single fluid description of a plasma that combines the motion of the ions and electrons.
2. MHD is the most simple, self-consistent description of plasma dynamics
3. MHD is the most widely used plasma description used in space and astrophysics, chosen often for its simplicity.
4. Describes macroscopic dynamics of a plasma, at large length scales and long time scales.

A. The MHD Approximation

1. As usual consider characteristic system size and observation time

$$\left.\begin{array}{l} L \equiv \text{System Size} \\ \tau \equiv \text{observation time} \end{array}\right\} \text{Characteristic velocity } V_0 \equiv \frac{L}{\tau}$$

2. MHD Equations are valid under the following conditions:

a. Strong collisionality, $\lambda_{mfp} \ll L$ or $\tau \gg \frac{1}{\nu_{ei}}$

$$\left.\begin{array}{ll} \text{b. Non-relativistic,} & V_0^2/c^2 \ll 1 \\ \text{c. Magnetized,} & r_{Li} \ll L \end{array}\right\} \text{These imply Quasi-neutrality, } \sum_s n_s q_s = \rho_q = 0$$

B. Derivation of MHD Equations

1. For the conditions above, MHD is a rigorous limit of kinetic theory.
2. The MHD Equations are derived from moments of the Boltzmann Equation combined with Maxwell's Equations

3.a. Define: Mass Density $\boxed{\rho \equiv \sum_s n_s m_s}$

b. Define: Fluid Velocity $\boxed{\underset{\sim}{U} \equiv \frac{1}{\rho} \sum_s n_s m_s \underset{\sim}{U_s}}$ ← Mass-weighted fluid velocity (dominated by ion motion)

I. B. (Continued)

4. <u>Continuity Equation</u>: Sum of Zeroth Moment Equations over species

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underset{\sim}{U}) = 0}$$

5. <u>Momentum Equation</u>: Sum of First Moment Equations over Species

a. When summed over species, collisions have no contribution due to conservation of momentum

$$\boxed{\frac{\partial}{\partial t}(\rho\underset{\sim}{U}) + \nabla\cdot(\rho\underset{\sim}{U}\underset{\sim}{U}) = -\nabla\cdot\underset{\approx}{P} + \underset{\sim}{j}\times\underset{\sim}{B}}$$

b. To lowest order in $\frac{\lambda_{mfp}}{L}$, viscosity (within pressure tensor) may be neglected and pressure assumed to be isotropic, $\nabla\cdot\underset{\approx}{P} = \nabla p$

c. Using the continuity equation to simplify, we obtain

$$\boxed{\rho\frac{\partial \underset{\sim}{U}}{\partial t} + \rho(\underset{\sim}{U}\cdot\nabla)\underset{\sim}{U} = -\nabla p + \underset{\sim}{j}\times\underset{\sim}{B}}$$

6. <u>Ohm's Law</u>: Difference of First Moment Equations

a. Since $\underset{\sim}{j} \equiv \sum n_s q_s \underset{\sim}{U}_s = n_i e \underset{\sim}{U}_i - n_e e \underset{\sim}{U}_e = n_0 e(\underset{\sim}{U}_i - \underset{\sim}{U}_e)$

$$\boxed{\underset{\sim}{E} + \underset{\sim}{U}\times\underset{\sim}{B} = \eta \underset{\sim}{j}}$$

b. <u>Define</u>: Resistivity $\boxed{\eta = \frac{m_e \nu_{ei}}{e^2 n_0}}$ ← depends on electron-ion collision frequency, ν_{ei}

7. <u>Faraday's Law</u>: $\boxed{\frac{\partial \underset{\sim}{B}}{\partial t} = -\nabla\times\underset{\sim}{E}}$

8. <u>Ampere's Law</u>: In non-relativistic limit $\frac{V_0^2}{c^2} \ll 1$, drop Displacement current,

$$\boxed{\nabla\times\underset{\sim}{B} = \mu_0 \underset{\sim}{j}}$$

I. B. (Continued)

9. Gauss' Law: $\boxed{\nabla \cdot \underset{\sim}{E} = 0}$ (Charge density $\rho_q = 0$)

10. Zero Magnetic Divergence: $\boxed{\nabla \cdot \underset{\sim}{B} = 0}$

11. Adiabatic Equation of State:

a. For strongly collisional conditions this is the appropriate choice,

$$\boxed{\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0} \qquad \gamma = 5/3$$

C. Ideal MHD Equations:

1. Ideal limit takes viscosity & resistivity to be zero.
2. Ampere's Law used to eliminate $\underset{\sim}{j}$
3. Ohm's Law & Faraday's Law combined to eliminate $\underset{\sim}{E}$

Continuity Eq: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{U}) = 0$

Momentum Eq: $\rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho \underset{\sim}{U} \cdot \nabla \underset{\sim}{U} = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{(\underset{\sim}{B} \cdot \nabla)\underset{\sim}{B}}{\mu_0}$

Induction Eq: $\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B})$

Adiabatic Eq. of State: $\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0$

Closed Set of 8 equations for 8 unknowns: $\rho, \underset{\sim}{U}, \underset{\sim}{B}, p$

D. Resistive MHD Equations

1. The ideal limit is the lowest order description, neglecting dissipation.
2. To higher order, dissipative terms appear in the momentum equation (viscosity) & induction equation (resistivity)

3. a. $$\rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho \underset{\sim}{U} \cdot \nabla \underset{\sim}{U} = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{(\underset{\sim}{B} \cdot \nabla)\underset{\sim}{B}}{\mu_0} + \mu \nabla^2 \underset{\sim}{U}$$

b. Define: Coefficient of Shear Viscosity, μ

Lecture #5 (Continued) Howes (4)

I. D3. (Continued)

c. Momentum Eq. with Viscosity
$$\boxed{\frac{\partial \underset{\sim}{U}}{\partial t} + (\underset{\sim}{U}\cdot\nabla)\underset{\sim}{U} = -\frac{1}{\rho}\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{(\underset{\sim}{B}\cdot\nabla)\underset{\sim}{B}}{\mu_0 \rho} + \nu \nabla^2 \underset{\sim}{U}}$$

d. Define: $\boxed{\text{Kinematic viscosity, } \nu \equiv \frac{\mu}{\rho}}$

4. Induction Eq. with resistivity
$$\boxed{\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla\times(\underset{\sim}{U}\times\underset{\sim}{B}) + \frac{\eta}{\mu_0}\nabla^2 \underset{\sim}{B}}$$

II. Dimensionless Numbers in Fluid Dynamics:

A. General: 1. In fluid dynamics, dimensionless numbers characteristic of the flow enable quantitative characterization of different physical systems.

2. The dimensionless number typically characterizes the magnitude of the ratio of two terms in the dynamical equations.

B. The Reynolds Number

1. Ratio of convective to dissipative (viscous) term in momentum eq.
$$\frac{\partial \underset{\sim}{U}}{\partial t} + \underbrace{(\underset{\sim}{U}\cdot\nabla)\underset{\sim}{U}}_{\substack{\text{Convective term}\\ \text{(non-linear)}}} = -\frac{1}{\rho}\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{(\underset{\sim}{B}\cdot\nabla)\underset{\sim}{B}}{2\mu_0 \rho} + \underbrace{\nu \nabla^2 \underset{\sim}{U}}_{\text{viscous term}}$$

2. $$Re \equiv \frac{|(\underset{\sim}{U}\cdot\nabla)\underset{\sim}{U}|}{|\nu\nabla^2 \underset{\sim}{U}|} \sim \frac{(V_0^2/L)}{(\nu V_0/L^2)} \sim \frac{L V_0}{\nu} \Rightarrow \boxed{Re \equiv \frac{L V_0}{\nu}}$$

3. Reynolds number determines (empirically) whether flow is laminar or turbulent.

Low Re flows ($Re <$ few hundred) $\Rightarrow$ laminar

High Re flows ($Re \gtrsim 10^3$) $\Rightarrow$ turbulent

4. Most space & astrophysical plasma flows (because of large length scales and low viscosity) have very high Re ($\gtrsim 10^6$)

$\Rightarrow$ $\boxed{\text{TURBULENCE is ubiquitous in space \& astrophysical plasmas}}$

II. (Continued)

C. Magnetic Reynold's Number

1. Ratio of convective to dissipative (resistive) terms in induction eq.

$$\frac{\partial \underline{B}}{\partial t} = \underbrace{\nabla \times (\underset{\sim}{U} \times \underline{B})}_{\text{convective term}} + \underbrace{\frac{\eta}{\mu_0} \nabla^2 \underset{\sim}{B}}_{\text{resistive term}}$$

2. $$R_{eM} \equiv \frac{|\nabla \times (\underset{\sim}{U} \times \underline{B})|}{|\frac{\eta}{\mu_0} \nabla^2 \underline{B}|} \sim \frac{(V_0 B / L)}{\left(\frac{\eta B}{\mu_0 L^2}\right)} \sim \frac{\mu_0 L V_0}{\eta} \qquad \boxed{R_{eM} \equiv \frac{\mu_0 L V_0}{\eta}}$$

3a. In the limit $R_{eM} \gg 1$, convective terms dominates and resistivity can be ignored $\Rightarrow$ Ideal MHD!

b. In the limit $R_{eM} \ll 1$, resistive term dominates

4. Most space & astrophysical plasma flows (because of large length scales and low resistivity) have very high R_{eM} ($\gtrsim 10^6$)

5. Note that the equations of Ideal MHD are valid for $R_e \gg 1$ and $R_{eM} \gg 1$.

$\Rightarrow$ Ideal MHD is often a reasonable description for space & astrophysical plasmas

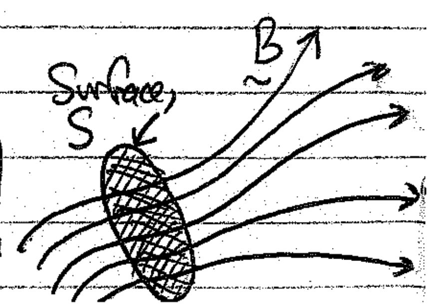
III. Frozen-In Flux

A. Theorems:

1. Several powerful theorems can be proven for plasmas with $R_{eM} \gg 1$, satisfying the Ideal MHD Induction Eq,

$$\frac{\partial \underline{B}}{\partial t} - \nabla \times (\underset{\sim}{U} \times \underline{B}) = 0$$

2. Define: Magnetic Flux $\boxed{\Phi_B \equiv \int_S \underline{B} \cdot d\underset{\sim}{A}}$



III. A. (Continued)

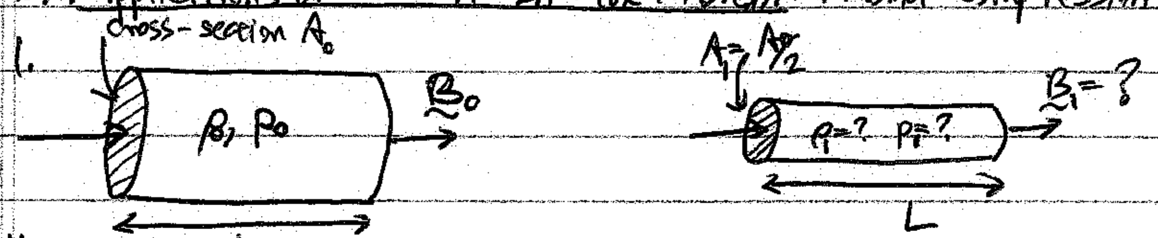
3. **The Frozen-In Flux Theorem:** The Magnetic flux through a surface moving with the plasma (at fluid velocity $\underset{\sim}{U}(x,t)$) remains constant.

4. **Theorem II:** The Magnetic field lines are frozen to the plasma/fluid flow.

5. Note: As long as $R_{eM} \gg 1$ $(\eta \to 0)$, these theorems apply whether or not the plasma is strongly collisional.
$\Rightarrow$ In collisionless plasmas (at scales $L \gg r_{Li}$), the magnetic flux is frozen-in to the plasma flow.

B. Applications of Frozen-In Flux Theorem: Plasma Compression or Expansion

1.



Magnetic Field

2. By Frozen in Theorem, $\Phi_0 = \Phi_1 \Rightarrow B_0 A_0 = B_1 A_1 \to B_1 = B_0 \frac{A_0}{A_1} = 2B_0$

3. Density: $\rho = \frac{mN}{V}$, $m = \text{const}$, $N = \text{const}$, $V_0 = A_0 L$ $\quad V_1 = A_1 L = \frac{A_0 L}{2} = V_0/2$

$$\rho_1 = \frac{mN}{V_1} = \frac{mN}{(V_0/2)} = 2\rho_0$$

4. Pressure: $\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = \text{const.} \Rightarrow \frac{p_0}{\rho_0^\gamma} = \frac{p_1}{\rho_1^\gamma} \Rightarrow p_1 = p_0\left(\frac{\rho_1}{\rho_0}\right)^\gamma = p_0\left(\frac{2\rho_0}{\rho_0}\right)^\gamma$

Thus, $p_1 = 2^\gamma p_0 = 2^{5/3} p_0 \simeq 3.17 p_0$

5. NOTE: a. For a collisionless plasma, use Double Adiabatic (CGL) Equations of State & Frozen-In Flux Theorem.
b. Can connect temperature to pressure using $p_s = n_s T_s$.

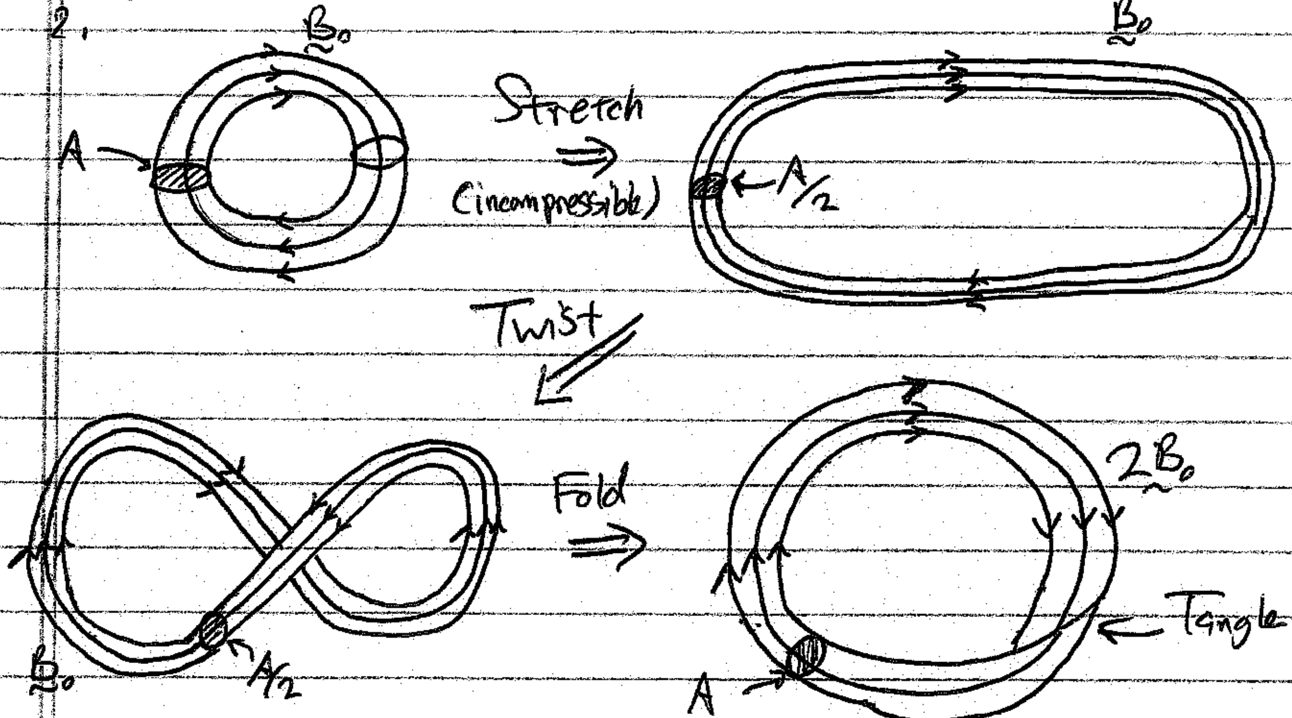
Lecture #5 (Continued) Hawes ⑦

III. (Continued)

C. Zeldovitch's Rope Dynamo

1. Dynamo theory is used to understand the generation of magnetic fields in space & astrophysical plasmas.

2.



3. Nearly the same (after Stretch-Twist-Fold) as initial configuration, but $\underset{\sim}{B}$ has doubled in strength, and there is a small tangle!

4. Energy required to stretch magnetic field leads to amplified $\underset{\sim}{B}$ field.

5. Such simple models can be used to attempt to explain the origin of astrophysical magnetic fields (alpha-omega dynamo, etc.)

IV. Magnetic Diffusion

A.1. In the limit $R_{eM} \ll 1$, the convection term in the induction eq. may be dropped, leading to a diffusion equation,

$$\boxed{\frac{\partial \underset{\sim}{B}}{\partial t} = \frac{\eta}{\mu_0} \nabla^2 \underset{\sim}{B}}$$

Lecture #5 (Continued)

IV. A. (Continued)

2. The timescale for the diffusion of the magnetic field, τ_{diff}, over a length scale L can be estimated by

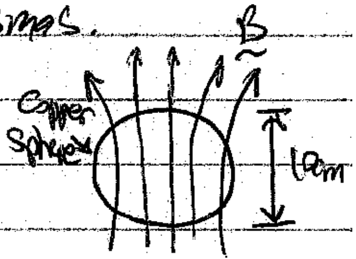
$$\frac{B}{\tau_{diff}} \sim \frac{\eta B}{\mu_0 L^2} \Rightarrow \boxed{\tau_{diff} \simeq \frac{\mu_0 L^2}{\eta}}$$

3. Using the definition for ν_{ei}, we can compute η

$$\eta = \frac{m_e \nu_{ei}}{e^2 n_0} = \frac{e^2 m_e^{1/2} \ln N_D}{2^{5/2} \pi \epsilon_0^2 T_e^{3/2}}$$

To find characteristic diffusion time in typical plasmas.

4. Example: Copper sphere of diameter 10cm



a. Resistivity of copper, $\eta = 1.7 \times 10^{-8}\ \Omega \cdot m$

b. $\tau_{diff} = \frac{\mu_0 L^2}{\eta} = \frac{(4\pi \times 10^{-7} \frac{H}{m})(0.1m)^2}{1.7 \times 10^{-8}\ \Omega \cdot m} = 0.7s$

5. For typical plasmas,

Plasma	$n(m^{-3})$	$T(K)$	$B(T)$	$L(m)$	$\nu_{ei}(s^{-1})$	$\eta(\Omega \cdot m)$	τ_{diff}
LAPD	10^{18}	10^5	0.06	0.4	3×10^6	10^{-4}	$1.7\times10^{-3}s$
Fusion Plasma	10^{21}	10^8	10	2.0	2×10^5	6×10^{-9}	$8\times10^2 s = 13m$
Solar Wind	10^7	10^5	10^{-8}	$1AU = 1.5\times10^{11}$	7×10^{-5}	2.5×10^{-4}	$5\times10^{19}s = 10^{12}y$
ISM	10^6	10^4	10^{-10}	$1pc = 3\times10^{16}m$	2×10^{-4}	7×10^{-3}	$2\times10^{29}s = 10^{21}y$

a. Although resistivities are larger than copper (except for fusion plasma), diffusion times are long because of the scale of the plasma

5. Earth's molten core has $\tau_{diff} \sim 10^4$ years

$\Rightarrow$ Earth's magnetic field <u>must</u> be maintained by some dynamo process!

6. For most space and astrophysical plasmas, ideal MHD is a good approximation for dynamics at large scales $L \gg r_{Li}$ (however, in collisionless conditions, parallel motions may require a kinetic treatment).