Lecture #6 MHD Waves

I. Ideal MHD Equations

compression — Thermal Pressure — Magnetic Pressure — Magnetic Tension

Continuity Eq.	$\frac{\partial \rho}{\partial t} + (\mathbf{U}\cdot\nabla)\rho = -\rho(\nabla\cdot\mathbf{U})$
Momentum Eq.	$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{U} = -\frac{1}{\rho}\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{\mu_0 \rho}$
Induction Eq.	$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{B} = -\mathbf{B}(\nabla\cdot\mathbf{U}) + (\mathbf{B}\cdot\nabla)\mathbf{U}$
Adiabatic Eq. of State	$\frac{\partial p}{\partial t} + (\mathbf{U}\cdot\nabla)p = -\gamma p(\nabla\cdot\mathbf{U})$

Stretching of Magnetic Field lines

$\nabla\cdot\mathbf{U}$ is Compression

1. Here we have used the definition of the convective derivative

$$\boxed{\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla}$$

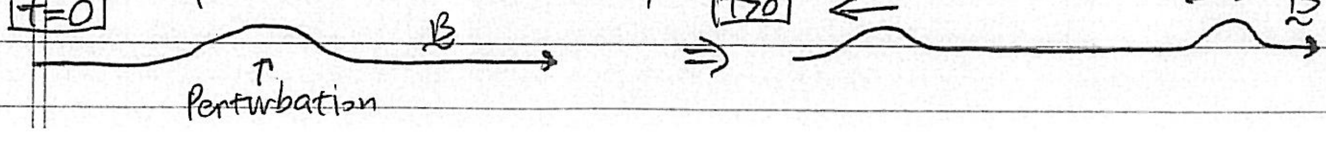
and the continuity equation to simplify the adiabatic eq. of state.

2. We have used vector identities [NRL p.4 (10)] and $\nabla\cdot\mathbf{B} = 0$ to simplify Induction eq.

II. Linear Wave Dispersion Relation

A. Linear Wave Modes

1. The linear wave modes of plasma describe the characteristic response of the plasma to small amplitude perturbations.
2. A general perturbation (of small amplitude) can be decomposed into its component linear MHD wave modes — these waves carry the disturbance away as waves.



II. A. (Continued)

3. Linear Dispersion Relation

a. The linear dispersion contains all of the information about the characteristic linear wave modes.

b. IMPORTANT: The technique for determining the linear dispersion relation arises frequently in the study of plasmas!

B. General Procedure for Calculating the Linear Dispersion Relation

$\boxed{\text{1. Linearize the set of equations}}$

$\boxed{\text{2. Fourier transform in space and time}}$

a. Any disturbance can be written as a sum of Fourier modes:

$$\rho(\underset{\sim}{x},t) = \sum_{\underset{\sim}{k}} \rho(\underset{\sim}{k})\, e^{+i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega(\underset{\sim}{k})t}$$

sum over all possible wave vectors

This frequency $\omega(\underset{\sim}{k})$ is determined by the dispersion relation.

b. Since equations are now linear, we need only solve for a single (arbitrary) wavevector $\underset{\sim}{k}$

c. In practice, the Fourier transform simply replaces

$$\boxed{\frac{\partial}{\partial t} \rightarrow -i\omega} \qquad \boxed{\nabla \rightarrow i\underset{\sim}{k}}$$

d. These leads to an algebraic set of equations.

$\boxed{\text{3. Collect system of equations into } N\times N \text{ linear matrix equation.}}$

$$\begin{pmatrix} \text{N×N matrix} \\ \underset{\approx}{D} \end{pmatrix} \begin{pmatrix} \\ \\ \end{pmatrix} = 0$$

N×1 vector of variables (N=8 for MHD: $\rho, \underset{\sim}{U}, \underset{\sim}{B}, P$)

Lecture #6 (Continued)

II. B. (Continued)

$\boxed{\text{4. Solve for the determinant of } N \times N \text{ matrix} = 0}$

$$|\underset{\sim}{D}| = 0 \leftarrow \underline{\text{Dispersion relation}}$$

a. This leads to an expression $\omega = \omega(\underset{\sim}{k})$
↑ Solutions of dispersion relation.

C. Example of the Linear Dispersion Relation Calculation

1. Linearize: $\frac{\partial \underset{\sim}{B}}{\partial t} + (\underset{\sim}{U} \cdot \nabla)\underset{\sim}{B} = -\underset{\sim}{B}\, \nabla \cdot \underset{\sim}{U} + (\underset{\sim}{B} \cdot \nabla)\underset{\sim}{U}$

a. Take uniform $\underset{\sim}{B}_0 = B_0 \hat{\underset{\sim}{z}}$ in homogeneous plasma with no mean flow!

$$\left.\begin{aligned} \rho &= \rho_0 + \epsilon \rho_1 \\ \underset{\sim}{U} &= \epsilon \underset{\sim}{U}_1 \\ \underset{\sim}{B} &= \underset{\sim}{B}_0 + \epsilon \underset{\sim}{B}_1 \\ p &= p_0 + \epsilon p_1 \end{aligned}\right\}$$ where $\epsilon \ll 1$ and ρ_0, B_0, and p_0 are $\underline{\text{constants}}$ in space and time

b. Substitute into equation

$\underset{\sim}{B}_0$ constant

$$\cancelto{0}{\frac{\partial \underset{\sim}{B}_0}{\partial t}} + \epsilon \frac{\partial \underset{\sim}{B}_1}{\partial t} + \underbrace{\cancelto{0}{\epsilon \underset{\sim}{U}_1 \cdot \nabla \underset{\sim}{B}_0}}_{\underset{\sim}{B}_0 \text{ constant}} + \epsilon^2 \underset{\sim}{U}_1 \cdot \nabla \underset{\sim}{B}_1 = -\epsilon \underset{\sim}{B}_0\, \nabla \cdot \underset{\sim}{U}_1 - \epsilon^2 \underset{\sim}{B}_1\, \nabla \cdot \underset{\sim}{U}_1 + \epsilon\, \underset{\sim}{B}_0 \cdot \nabla \underset{\sim}{U}_1 + \epsilon^2 \underset{\sim}{B}_1 \cdot \nabla \underset{\sim}{U}_1$$

c. Drop all terms with ϵ^2 or higher $\Rightarrow$

$$\boxed{\frac{\partial \underset{\sim}{B}_1}{\partial t} = -\underset{\sim}{B}_0\, \nabla \cdot \underset{\sim}{U}_1 + \underset{\sim}{B}_0 \cdot \nabla \underset{\sim}{U}_1} \leftarrow \text{Linearized Induction Eq.}$$

2. Fourier Transform: $\frac{\partial}{\partial t} \rightarrow -i\omega \qquad \nabla \rightarrow i\underset{\sim}{k}$

a. $-i\omega \underset{\sim}{B}_1 = -i\, \underset{\sim}{B}_0\, \underset{\sim}{k} \cdot \underset{\sim}{U}_1 + i\, \underset{\sim}{B}_0 \cdot \underset{\sim}{k}\, \underset{\sim}{U}_1$

b. $\boxed{\omega \underset{\sim}{B}_1 = \underset{\sim}{B}_0 (\underset{\sim}{k} \cdot \underset{\sim}{U}_1) - (\underset{\sim}{B}_0 \cdot \underset{\sim}{k})\, \underset{\sim}{U}_1}$

Lecture #6 (Continued) Howes ④

II. C2 (Continued)

c. Performing the Fourier Transform on all equations leads to

$$\boxed{\begin{aligned}
&① \ \omega \rho_1 = \rho_0 (\underset{\sim}{k} \cdot \underset{\sim}{U_1}) \\
&② \ \omega \underset{\sim}{U_1} = \underset{\sim}{k}\left(\frac{p_1}{\rho_0} + \frac{\underset{\sim}{B_0} \cdot \underset{\sim}{B_1}}{\mu_0 \rho_0}\right) - \frac{(\underset{\sim}{B_0} \cdot \underset{\sim}{k}) \underset{\sim}{B_1}}{\mu_0 \rho_0} \\
&③ \ \omega \underset{\sim}{B_1} = \underset{\sim}{B_0} (\underset{\sim}{k} \cdot \underset{\sim}{U_1}) - (\underset{\sim}{B_0} \cdot \underset{\sim}{k}) \underset{\sim}{U_1} \\
&④ \ \omega p_1 = \gamma p_0 (\underset{\sim}{k} \cdot \underset{\sim}{U_1})
\end{aligned}}$$

3. Collect system to N×N Matrix equation:

a. In general, MHD has 8 unknowns $(\rho, U_x, U_y, U_z, B_x, B_y, B_z, p)$

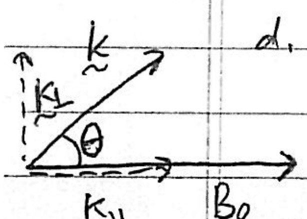
b. But, it is easier to compute the determinant of a 3×3 matrix

c. Thus, eliminate p_1, $\underset{\sim}{B_1}$, and ρ_1

1. Use ④ to eliminate p_1
2. Use ③ to eliminate $\underset{\sim}{B_1}$
3. Since ρ_1 doesn't appear in other equations, no need to do anything.

d. Without loss of generality, we take $\underset{\sim}{B_0} = B_0 \hat{\underset{\sim}{z}}$

and $\underset{\sim}{k} = k_\perp \hat{\underset{\sim}{x}} + k_\parallel \hat{\underset{\sim}{z}} = k \sin\theta \hat{\underset{\sim}{x}} + k \cos\theta \hat{\underset{\sim}{z}}$



e. Finally, we obtain the 3×3 Matrix equation for $(U_x, U_y, U_z) = \underset{\sim}{U_1}$

$$\boxed{\begin{pmatrix} \omega^2 - k^2 \sin^2\theta (c_s^2 + v_A^2) - k^2 \cos^2\theta \, v_A^2 & 0 & -k^2 \sin\theta \cos\theta \, c_s^2 \\ 0 & \omega^2 - k^2 \cos^2\theta \, v_A^2 & 0 \\ -k^2 \sin\theta \cos\theta \, c_s^2 & 0 & \omega^2 - k^2 \cos^2\theta \, c_s^2 \end{pmatrix} \begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix} = 0}$$

where we have defined:

Define: $\boxed{\text{Alfven Speed} \quad v_A^2 \equiv \frac{B_0^2}{\mu_0 \rho_0}}$

Define: $\boxed{\text{Sound Speed} \quad c_s^2 = \frac{\gamma p_0}{\rho_0}}$

II. C. (Continued)

4. Solve for $|\underset{\sim}{D}| = 0$:

Not the same as the 3x3 Matrix $\underset{\sim}{D}$

a. At this step, we obtain the Dispersion Relation $D(\omega, \underset{\sim}{k}) = 0$.

b. We may then solve for the wave mode frequencies, $\omega = \omega(\underset{\sim}{k})$.

c. Computing the determinant and simplifying, we obtain:

$$\boxed{(\omega^2 - k^2\cos^2\theta\, V_A^2)\left[\omega^4 - \omega^2 k^2 (C_s^2 + V_A^2) + k^4 \cos^2\theta\, C_s^2 V_A^2\right] = 0}$$

IDEAL MHD DISPERSION RELATION

III. Linear MHD Wave Properties

A. General: 1. Six Solutions to $D(\omega, \underset{\sim}{k})$

a. Two Alfven waves

b. Two Fast waves

c. Two Slow waves

2. 3x3 Matrix Decouples into two systems:

$$\underbrace{(\omega^2 - k^2\cos^2 V_A^2)\, U_y = 0}_{\text{Alfven Waves}}$$

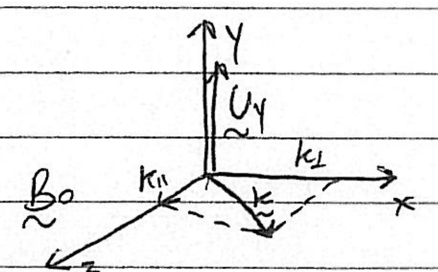
$U_y \neq 0$, $U_x = U_z = 0$

Typically most important wave mode in astrophysical plasmas.

$$\underbrace{\begin{pmatrix} \omega^2 - k^2\sin^2\theta(C_s^2 + V_A^2) + k^2\cos^2\theta\, V_A^2 & -k^2\sin\theta\cos\theta\, C_s^2 \\ -k^2\sin\theta\cos\theta\, C_s^2 & \omega^2 - k^2\cos^2\theta\, C_s^2 \end{pmatrix}\begin{pmatrix} U_x \\ U_z \end{pmatrix} = 0}_{\text{Fast and Slow Waves}}$$

$U_x \neq 0$, $U_z \neq 0$, $U_y = 0$

B. The Alfven Waves



1. Solution: $\boxed{\omega = \pm k_\parallel V_A}$

2. $U_y \neq 0 \Rightarrow$ Plasma fluid velocity out of plane containing $\underset{\sim}{k}$ & $\underset{\sim}{B_0}$

3. $U_z = U_x = 0$

4. Therefore, $\underset{\sim}{k} \cdot \underset{\sim}{U_1} = 0 \Rightarrow \nabla \cdot \underset{\sim}{U_1} = 0 \Rightarrow$ $\boxed{\text{Alfven wave has no compression!}}$

Lecture #6 (Continued)

III. B. (Continued)

5. What are the relevant linearized equations?

a. $\frac{\partial U_y}{\partial t} = \frac{B_0}{\mu_0 \rho_0} \frac{\partial B_y}{\partial z}$

b. $\frac{\partial B_y}{\partial t} = B_0 \frac{\partial U_y}{\partial z}$

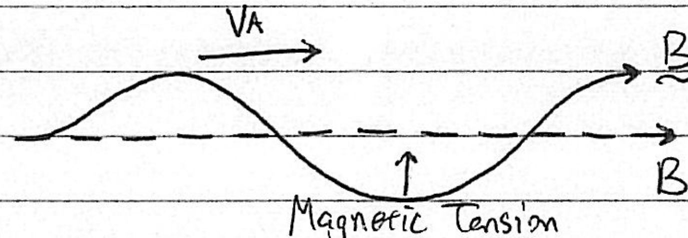
$\Rightarrow$ Wave equation: $\boxed{\frac{\partial^2 U_y}{\partial t^2} = V_A^2 \frac{\partial^2 U_y}{\partial z^2}}$

Restoring Force is Magnetic Tension

c. Since $\nabla \cdot \underset{\sim}{U}_1 = 0$, then $\rho_1 = 0$ and $p_1 = 0$

$\Rightarrow$ No pressure or density fluctuations

6. $\underset{\sim}{B} = \underset{\sim}{B}_0 + \underset{\sim}{B}_1 = B_0 \hat{z} + B_y \hat{y}$



a. Alfven wave is like a wave on a stretched rubber band.

b. Wave propagates at speed V_A along magnetic field $\underset{\sim}{B}_0$.

C. Fast and Slow Waves

1. Solution:

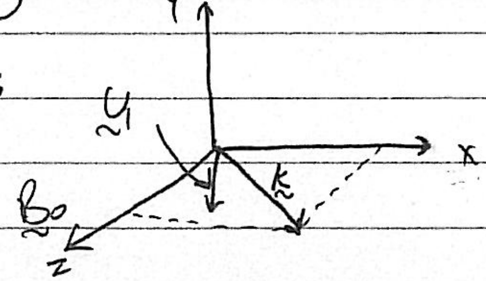
$$\boxed{\frac{\omega^2}{k^2} = \frac{1}{2}(C_s^2 + V_A^2) \pm \frac{1}{2}\sqrt{(C_s^2 + V_A^2)^2 - 4 C_s^2 V_A^2 \cos^2\theta}}$$

⊕ sign: Fast wave

⊖ sign: Slow wave

2. $U_x \neq 0$, $U_z \neq 0$, $U_y = 0$

Plasma Fluid velocity is in plane of $\underset{\sim}{k}$ & $\underset{\sim}{B}_0$



3. $\underset{\sim}{k} \cdot \underset{\sim}{U}_1 \neq 0 \Rightarrow \nabla \cdot \underset{\sim}{U}_1 \neq 0 \Rightarrow$ Fast & slow waves are compressible

4. Since $\nabla \cdot \underset{\sim}{U}_1 \neq 0$, then $\frac{\partial \rho_1}{\partial t} \neq 0$ and $\frac{\partial p_1}{\partial t} \neq 0$, so

Fast and slow waves have density and pressure fluctuations

Lecture #6 (Continued)

III. C. (Continued)

Greater "spring constant" $\Rightarrow$ faster wave

5. Restoring Forces:

a. Fast Wave: 1. Thermal and magnetic pressure act together
2. Magnetic Tension

b. Slow Wave: 1. Thermal and magnetic pressure oppose each other
2. Magnetic Tension

Lower "spring constant" $\Rightarrow$ slower wave.

D. The Entropy Mode:

1. MHD Equations have 8 equations & 8 unknowns
$\Rightarrow$ Why do we only have 6 solutions?

2. Two Reasons:

a. $\nabla \cdot \underline{B} = 0$: This is an additional constraint, $k_\perp B_x + k_\parallel B_z = 0$.
This eliminates one unknown, leaving 7 unknowns!

b. A more careful treatment allows fluctuations of ρ_1 and p_1 that satisfy specific entropy $\boxed{S \equiv C \frac{p}{\rho^\gamma} = \text{const.}}$

These entropy conserving fluctuations have $\omega = 0$, and have been dropped in our dispersion relation analysis.

3. So, there does exist a $\boxed{7^{th},\ \omega = 0 \text{ Entropy Mode}}$ in addition to the 6 waves, ($\pm$ Fast, $\pm$ Alfven, $\pm$ Slow).

IV. More Properties of MHD

A. Dimensionless Dispersion Relation:

1. Define: $\bar{\omega} \equiv \frac{\omega}{k v_A}$ Define: Plasma Beta $\beta = \frac{c_s^2}{v_A^2}$

2. $$\boxed{(\bar{\omega}^2 - \cos^2\theta)\left[\bar{\omega}^4 - \bar{\omega}^2(1+\beta) + \beta\cos^2\theta\right] = 0}$$

3. Demonstrates that the MHD Dispersion relation depends only on two parameters! $\boxed{\bar{\omega} = \bar{\omega}(\beta, \theta)}$

Lecture #6 (Continued)

IV. (Continued)

B. Conservation of Energy in Ideal MHD:

1.
$$\frac{\partial}{\partial t}\Big(\underbrace{\frac{1}{2}\rho U^2}_{\substack{\text{Kinetic}\\ \text{Energy}}} + \underbrace{\frac{P}{\gamma-1}}_{\substack{\text{Internal}\\ \text{(Thermal)}\\ \text{Energy}}} + \underbrace{\frac{B^2}{2\mu_0}}_{\substack{\text{Magnetic}\\ \text{Energy}}}\Big) + \nabla\cdot\Big(\underbrace{\frac{1}{2}\rho U^2 \mathbf{U}}_{\substack{\text{Flux of}\\ \text{Kinetic}\\ \text{Energy}}} + \underbrace{\frac{\gamma P}{\gamma-1}\mathbf{U}}_{\substack{\text{Enthalpy}\\ \text{Flux}}} + \underbrace{\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}}_{\substack{\text{Poynting}\\ \text{Flux}}}\Big) = 0$$

2. Conserved Energy: $\boxed{E = \int d^3x \left[\frac{1}{2}\rho U^2 + \frac{P}{\gamma-1} + \frac{B_0^2}{2\mu_0}\right]}$

C. MHD Linear Eigenfunctions:

1. Each wave mode has a particular eigenfunction $(\rho_1, \mathbf{U}_1, \mathbf{B}_1, P_1)$
   a. Example: Alfven Wave: $\rho_1 = 0$, $P_1 = 0$, $U_x = 0$, $U_z = 0$, $B_x = 0$, $B_z = 0$
$$\frac{B_y}{B_0} = \pm\frac{U_y}{v_A}$$

2. How do we determine this?
   a. Go back to 3x3 Matrix Equation for MHD Dispersion Relation.
   b. Choose a value for one component, for example, $U_y = U_0$.
   c. Solve for all other quantities.