Lecture #9 More About MHD Waves

Howes ①

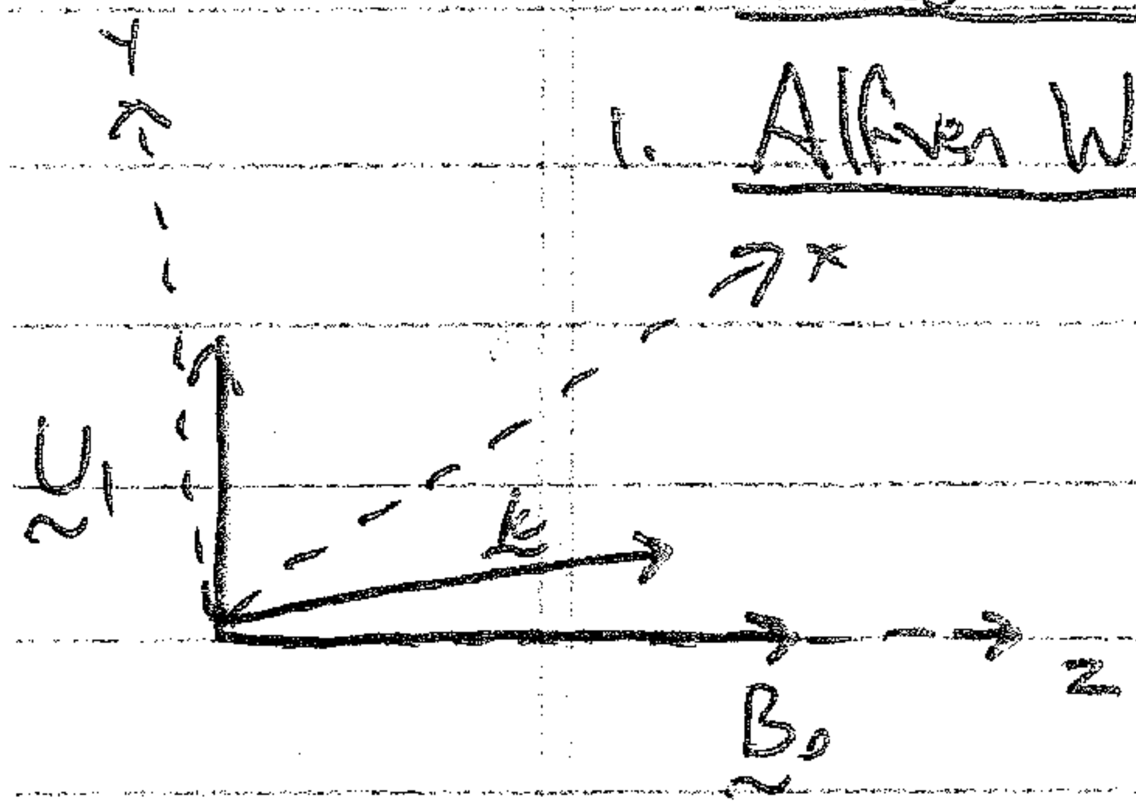
I. Review

A. Last time, we linearized the MHD equations, assumed plane wave (Fourier) solutions, and solved to obtain the MHD Dispersion Relation:

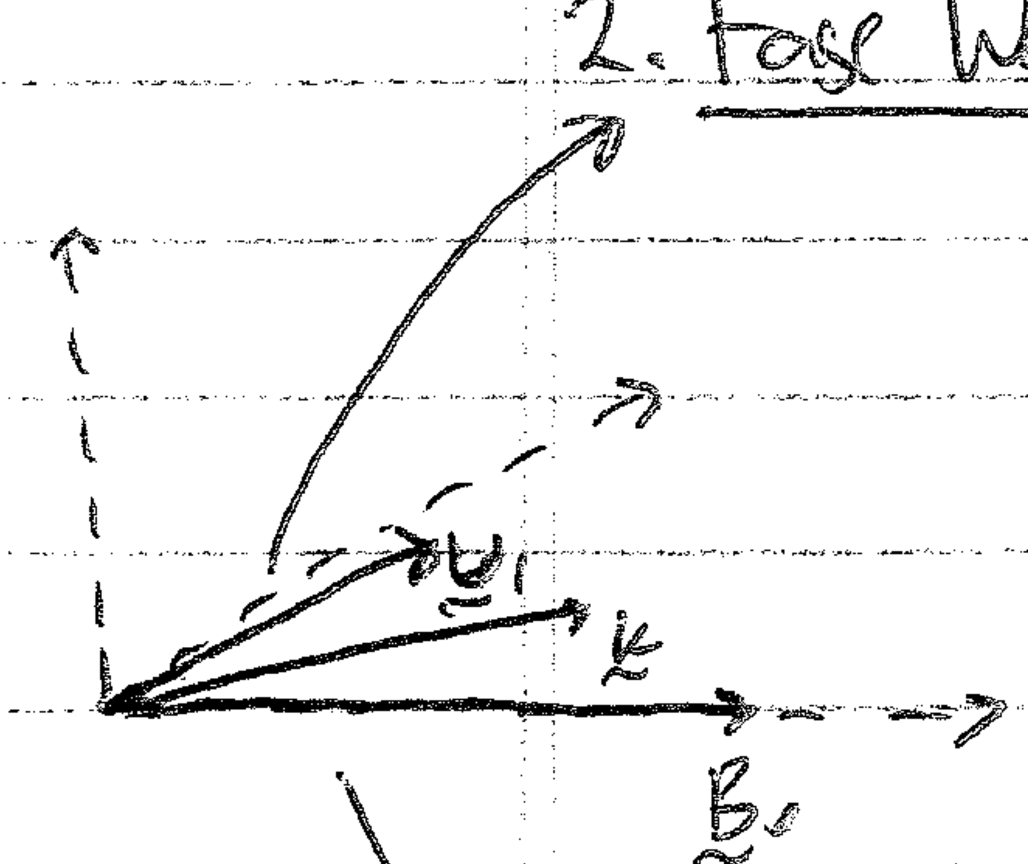
$$(\omega^2 - k^2\cos^2\theta\, V_A^2)\left[\omega^4 - \omega^2 k^2 (C_s^2 + V_A^2) + k^4 \cos^2\theta\, C_s^2 V_A^2\right] = 0$$

where $\underset{\sim}{k}$ at angle θ to $\underset{\sim}{B}_0$: $\underset{\sim}{B}_0 \cdot \underset{\sim}{k} = B_0 k \cos\theta$

B. Three Wave Modes:



1. Alfven Waves:
   a. $\omega^2 = k_\parallel^2 V_A^2$
   b. Motion out of the plane defined by $\underset{\sim}{B}_0$, $\underset{\sim}{k}$
   c. Incompressible
   d. Restoring Force: Magnetic Tension alone



2. Fast Waves:
   a. $\frac{\omega^2}{k^2} = \frac{1}{2}(C_s^2 + V_A^2) + \frac{1}{2}\sqrt{(C_s^2+V_A^2)^2 - 4C_s^2 V_A^2 \cos^2\theta}$
   b. Motion in the plane of $\underset{\sim}{B}_0$ and $\underset{\sim}{k}$
   c. Compressible (usually)
   d. Restoring Force: i) Thermal and Magnetic Pressure Add!
      ii) Magnetic Tension

3. Slow Waves:
   a. $\frac{\omega^2}{k^2} = \frac{1}{2}(C_s^2 + V_A^2) - \frac{1}{2}\sqrt{(C_s^2+V_A^2)^2 - 4C_s^2 V_A^2 \cos^2\theta}$
   b. Motion in the plane of $\underset{\sim}{B}_0$ and $\underset{\sim}{k}$
   c. Compressible
   d. Restoring Force i) Thermal and Magnetic Pressure Subtract!
      ii) Magnetic Tension

II. Polar Plot of MHD Wave Phase Speeds:

A. Dimensionless Version of MHD Dispersion Relation

1. Take $k_\parallel = k\cos\theta$ and $k_\perp = k\sin\theta$,
2. Normalize by dividing by ω_{ci}^6:

$$\left(\frac{\omega^2}{\omega_{ci}^2} - k_\parallel^2\frac{V_A^2}{\omega_{ci}^2}\right)\left[\frac{\omega^4}{\omega_{ci}^4} - \frac{\omega^2}{\omega_{ci}^2}(k_\perp^2+k_\parallel^2)\frac{V_A^2}{\omega_{ci}^2}\left(1+\frac{C_s^2}{V_A^2}\right) + k_\parallel^2(k_\perp^2+k_\parallel^2)\frac{V_A^4}{\omega_{ci}^4}\frac{C_s^2}{V_A^2}\right]=0$$

3. NOTE: a. Let $\tilde{\omega} = \frac{\omega}{\omega_{ci}}$

b. $$\frac{V_A^2}{\omega_{ci}^2} = \frac{\frac{B_0^2}{\mu_0\rho_0}}{\left(\frac{q_i^2B_0^2}{m_i^2}\right)} = \frac{1}{\mu_0\epsilon_0}\left(\frac{\epsilon_0 m_i}{n_0 q_i^2}\right) = \frac{c^2}{\omega_{pi}^2} \Rightarrow$$ This is the ion inertial length.

DEFINE: $d_i \equiv \frac{c}{\omega_{pi}} = \frac{V_A}{\omega_{ci}}$

c. $$\frac{C_s^2}{V_A^2} = \left(\frac{\gamma p_0}{\rho_0}\right)\left(\frac{\mu_0\rho_0}{B_0^2}\right) = \frac{\gamma}{2}\frac{2\mu_0 p_0}{B_0^2} = \frac{\gamma}{2}\beta$$ ← Plasma β: $=\frac{\text{Thermal Press}}{\text{Magnetic Press.}}$

$$\boxed{\beta = \frac{2\mu_0 p_0}{B_0^2}}$$

4. Thus,

$$\left(\tilde{\omega}^2 - k_\parallel^2 d_i^2\right)\left[\tilde{\omega}^4 - \tilde{\omega}^2\left(k_\perp^2 d_i^2 + k_\parallel^2 d_i^2\right)\left(1+\frac{\gamma}{2}\beta\right) + k_\parallel^2 d_i^2\left(k_\perp^2 d_i^2 + k_\parallel^2 d_i^2\right)\frac{\gamma}{2}\beta\right]=0$$

5. There are only three parameters (dimensionless) which $\tilde{\omega}$ depends on:

$$\boxed{\tilde{\omega} = \tilde{\omega}_{MHD}(k_\perp d_i, k_\parallel d_i, \beta)}$$

a. Two define the parallel & perpendicular components of the wavevector (this is characteristic of most dispersion relations)

b. Only one other dimensionless parameter: $\boxed{\beta}$

6. NOTE: a. $d_i = \frac{\rho_i}{\sqrt{\beta_i}}$ where $\beta_i = \frac{2\mu_0 p_i}{B_0^2} = \frac{\beta}{2}$ for $T_i = T_e$ (true for MHD)

b. Thus, we could write $\tilde{\omega} = \tilde{\omega}_{MHD}(k_\perp\rho_i, k_\parallel\rho_i, \beta_i)$

Lecture #9 (Continued)

II. A. (Continued)

7. Validity of MHD Approximation:

a. Remember $r_{Li} \ll L$, so if $L \sim \frac{1}{k}$, this means $\boxed{k r_{Li} \ll 1}$

b. Also

i. $V_0 = \frac{L}{T} \Rightarrow r_{Li} \ll L = T V_0$

ii. For $V_0 \sim V_{ti}$ and using $r_{Li} = \frac{v_{ti}}{\omega_{ci}}$, we get $\frac{\cancel{v_{ti}}}{\omega_{ci}} \ll \tau \cancel{v_{ti}}$

iii. Take $\omega \sim \frac{1}{T}$, giving us $\boxed{\omega \ll \omega_{ci}}$

c. Thus $\tilde{\omega} = \tilde{\omega}_{MHD}(k_\perp r_{Li}, k_\parallel r_{Li}, \beta_i)$ is valid when $\tilde{\omega} \ll 1$

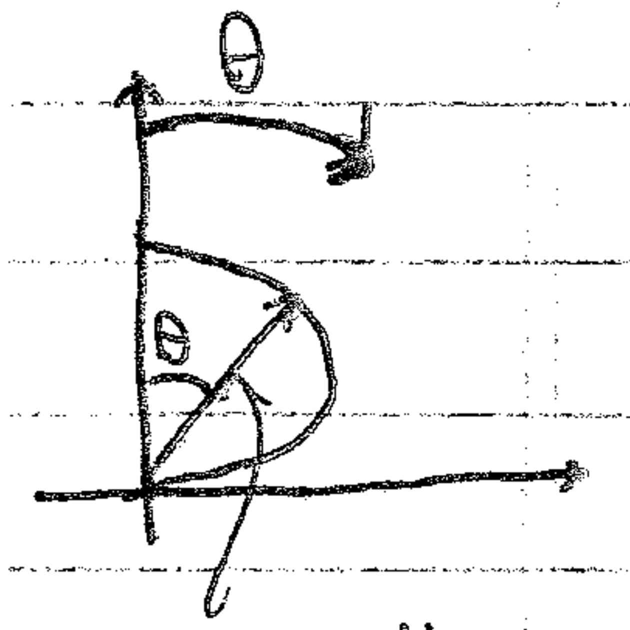
$k_\perp r_{Li}, k_\parallel r_{Li} \ll 1.$

B. Limits of $\frac{\omega}{k}$ at $\theta = 0.$

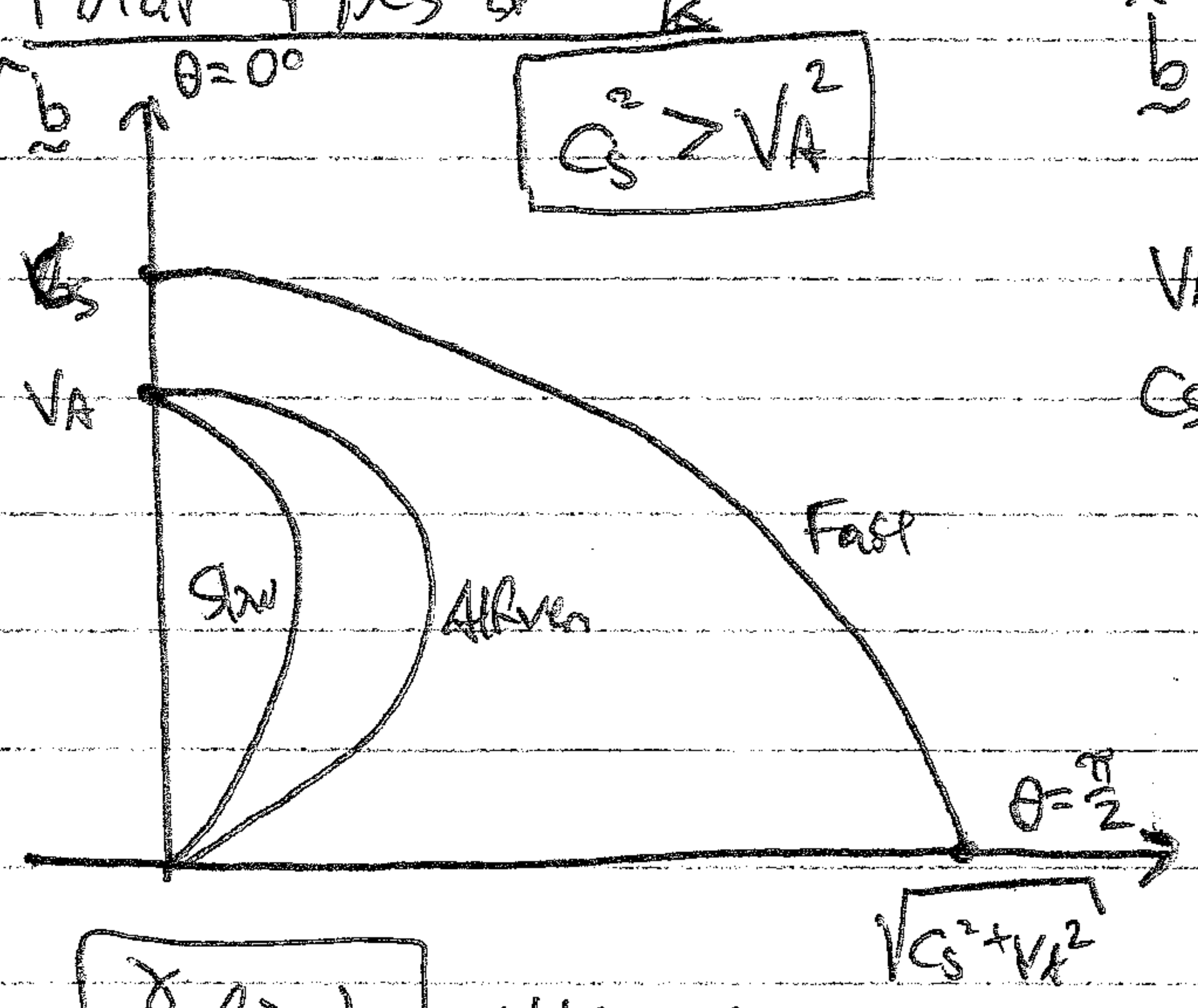
1. Phase velocity $V_p = \frac{\omega}{k}$ for waves at $\theta = 0$

	When $C_s^2 > V_A^2$:	When $C_s^2 < V_A^2$:
Fast	$\frac{\omega}{k} = C_s^2$	$\frac{\omega}{k} = V_A^2$
Alfven	$\frac{\omega}{k} = V_A^2$	$\frac{\omega}{k} = V_A^2$
Slow	$\frac{\omega}{k} = V_A^2$	$\frac{\omega}{k} = C_s^2$

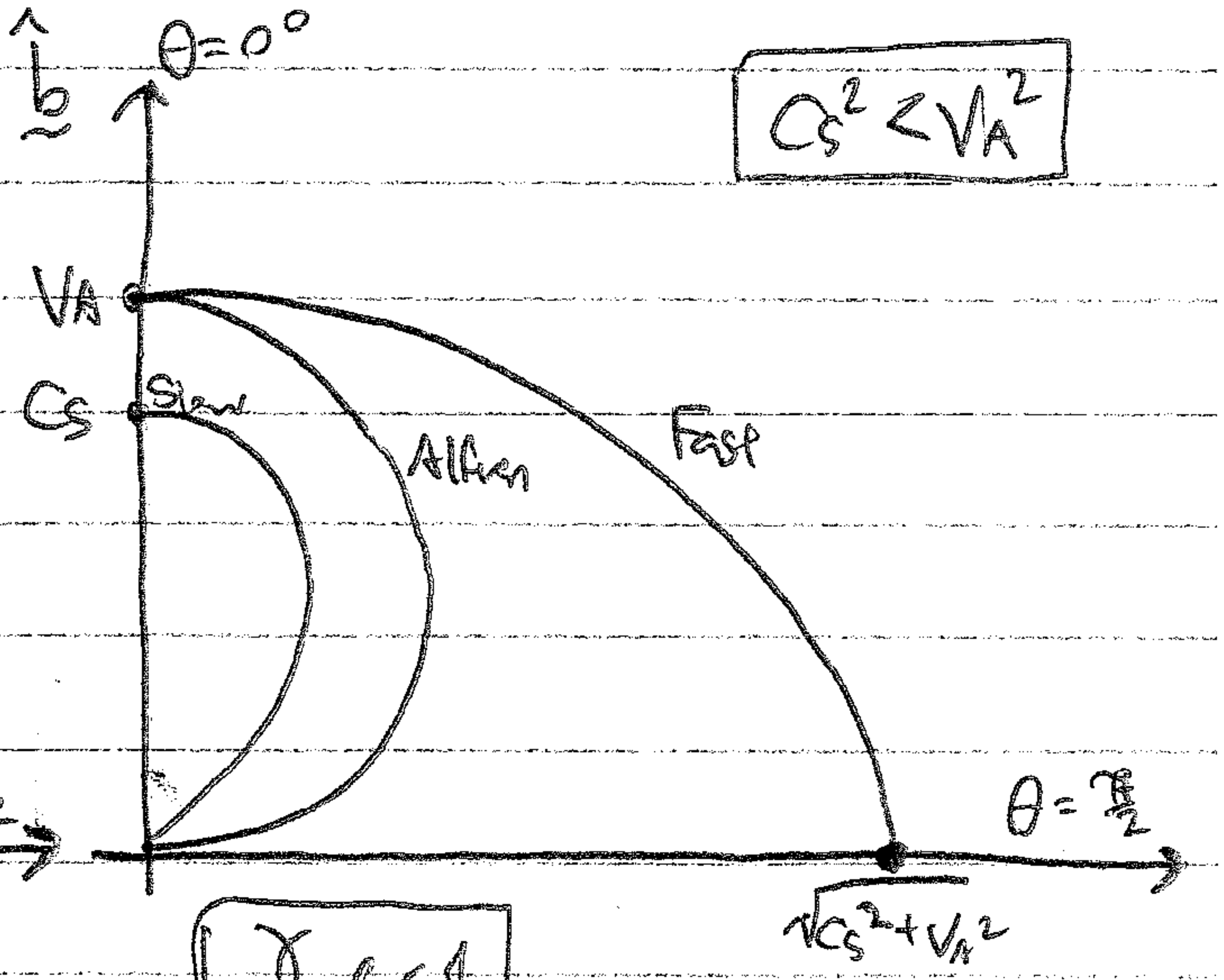
C. Polar Plots of $\frac{\omega}{k}$:



$\frac{\omega}{k}$ is radius



$\boxed{\frac{\gamma}{2}\beta > 1}$ HIGH BETA



$\boxed{\frac{\gamma}{2}\beta < 1}$ LOW BETA

III. Conservation of Energy in Ideal MHD:

A.1. The Ideal MHD Equations can be manipulated to give a law for the Conservation of Energy:

$$\frac{\partial}{\partial t}\Big(\underbrace{\tfrac{1}{2}\rho U^2}_{\text{Kinetic Energy}} + \underbrace{\frac{P}{\gamma-1}}_{\text{Internal (Thermal) Energy}} + \underbrace{\frac{B^2}{2\mu_0}}_{\text{Magnetic Energy}}\Big) + \nabla\cdot\Big(\underbrace{\tfrac{1}{2}\rho U^2 \underset{\sim}{U}}_{\text{Flux of Kinetic Energy}} + \underbrace{\frac{\gamma P}{\gamma-1}\underset{\sim}{U}}_{\text{Enthalpy Flux}} + \underbrace{\frac{1}{\mu_0}\underset{\sim}{E}\times\underset{\sim}{B}}_{\text{Poynting Flux}}\Big) = 0$$

2a. Integrating over all space, the volume integral of 2nd term can be converted to a surface integral by divergence theorem.

NRL p.5 (28)

b. For surface at infinity, you get

$$\frac{\partial E}{\partial t} = 0 \quad \text{with} \quad \boxed{E = \frac{1}{2}\rho U^2 + \frac{P}{\gamma-1} + \frac{B^2}{2\mu_0}}$$

Conserved Energy in Ideal MHD.

IV. The Entropy Mode:

A.1. The MHD Equations give **8** equations for 8 unknowns: $\rho, \underset{\sim}{U}, \underset{\sim}{B}, P$.

2. But, we found only 6 solutions to the dispersion relation.

3. In fact, a more careful analysis give two modes with $\omega = 0$. What do these modes correspond to?

B. Divergencelessness of $\underset{\sim}{B}$:

1. Remember, we must always satisfy $\nabla\cdot\underset{\sim}{B} = 0$, so there is really an additional constraint, so we only have 7 unknowns, and thus seven solutions.

C. The Entropy Mode:

1. We define DEF: Specific Entropy $\boxed{S = C\frac{P}{\rho^\gamma}}$ where C is some constant.

IV C (Continued)

2. Thus, the Adiabatic Equation of State is $\frac{ds}{dt} = 0$,
   $\Rightarrow$ Thus, entropy is conserved by these adiabatic fluctuations.

3. a. If we consider fluctuations, $\rho = \rho_0 + \rho_1$
   $s = s_0 + s_1$, etc.
   b. The other $\omega = 0$ mode is a zero frequency entropy mode.
   $s_1 \neq 0$, but $p_1 = 0$ (and so are $\underset{\sim}{U}_1 = 0$ & $\underset{\sim}{B}_1 = 0$).

4. Consider the ideal gas law: $pV = NkT$, or $p = nkT = \frac{\rho k T}{m}$
   a. We can have $p_1 = 0$ if $\rho_1 T_1 = \text{const.}$
   b. Thus density & temperature can vary to give constant pressure, $p_1 = 0$.

5. $\boxed{\text{The existence of the (oft-neglected) Entropy mode should not be forgotten}}$

6. There are $\underline{\underline{7}}$ solutions to ideal MHD dispersion relation
   a. Six waves ($\pm$ Fast, $\pm$ Alfven, $\pm$ Slow)
   b. One zero-frequency entropy mode

V. Eigenfunctions of the MHD Eigenmodes

A. How do we determine eigenfunctions ($\rho_1, \underset{\sim}{U}_1, \underset{\sim}{B}_1, p_1$) for a given wave mode?
   1. We must go back to the simplified matrix equation for MHD.
   2. Choose a value for one component.
   3. Solve for all other quantities.

Lecture #9 (Continued) Howes ⑥

V. (Continued)

B. Example: Eigenfunctions for $k_\parallel = k_\perp = k_0$ ($\theta = 45°$)

1. In this case, the vector equation for $\underset{\sim}{U}$ is

$$\begin{pmatrix} \omega^2 - k_0^2(c_s^2 + 2v_A^2) & 0 & -k_0^2 c_s^2 \\ 0 & \omega^2 - k_0^2 v_A^2 & 0 \\ -k_0^2 c_s^2 & 0 & \omega^2 - k_0^2 c_s^2 \end{pmatrix} \begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix} = 0$$

2. ~~[illegible]~~ As clear above, U_y is decoupled from U_x and U_z.

3. Let's find the ~~fast~~ slow wave eigenfunction for $U_x = U_0$ (Take $U_y = 0$)

a. $$\begin{pmatrix} \omega^2 - k_0^2(c_s^2 + 2v_A^2) & -k_0^2 c_s^2 \\ -k_0^2 c_s^2 & \omega^2 - k_0^2 c_s^2 \end{pmatrix} \begin{pmatrix} U_x \\ U_z \end{pmatrix} = 0$$

b. I can use either equation to solve for U_z as a func of U_x.

$$-k_0^2 c_s^2 U_x + (\omega^2 - k_0^2 c_s^2) U_z = 0$$

$$\Rightarrow \boxed{U_z = \frac{k_0^2 c_s^2}{\omega^2 - k_0^2 c_s^2} U_0}$$

$U_x = U_0$.

c. Here, for the ~~fast~~ slow wave

$$\omega^2 = \frac{1}{2} k_0^2 (c_s^2 + v_A^2) \pm \frac{k_0^2}{2} \sqrt{(c_s^2 + v_A^2)^2 - 2 c_s^2 v_A^2} = \frac{1}{2} k_0^2 (c_s^2 + v_A^2) \left[1 \pm \sqrt{1 - \frac{2 c_s^2 v_A^2}{(c_s^2 + v_A^2)^2}} \right]$$

i. ~~NOTE: [illegible]~~

So $$\boxed{\omega^2 = k_0^2 \frac{1}{2}(c_s^2 + v_A^2) \left[1 \pm \sqrt{1 - \frac{2 c_s^2 v_A^2}{(c_s^2 + v_A^2)^2}} \right]}$$

$+ \Rightarrow$ Fast

$- \Rightarrow$ Slow

4. Find density perturbation: $\omega \rho_1 = \rho_0 (\underset{\sim}{k} \cdot \underset{\sim}{U}_1) = \rho_0 (k_0 U_x + k_0 U_z)$

$$\rho_1 = \rho_0 \frac{k_0}{\omega}(U_x + U_z) = \rho_0 \frac{k_0}{\omega}\left(U_0 + \frac{k_0^2 c_s^2}{\omega^2 - k_0^2 c_s^2} U_0\right) = \rho_0 \frac{k_0}{\cancel{\omega}} \frac{\omega^{\cancel{2}}}{\omega^2 - k_0^2 c_s^2} U_x$$

$$\boxed{\rho_1 = \rho_0 \frac{\omega k_0}{\omega^2 - k_0^2 c_s^2} U_0}$$

V.B. (Continued)

5. Similarly $\boxed{p_1 = \gamma p_0 \frac{\omega k_0}{\omega^2 - k_0^2 c_s^2} U_0}$

6. Magnetic Field: $\omega \underset{\sim}{B_1} = \underset{\sim}{B_0}(\underset{\sim}{k} \cdot \underset{\sim}{U_1}) - (\underset{\sim}{B_0} \cdot \underset{\sim}{k}) \underset{\sim}{U_1}$

a. $\omega B_x = -B_0 k_0 U_x$ *

b. $\omega B_z = B_0(k_0 U_x + \cancel{k_0 U_z}) - \cancel{B_0 k_0 U_z} = B_0 k_0 U_x$

c. Thus
$$\boxed{\begin{aligned} B_x &= -B_0 \frac{k_0}{\omega} U_0 \\ B_z &= B_0 \frac{k_0}{\omega} U_0 \end{aligned}}$$

d. NOTE: $\nabla \cdot \underset{\sim}{B} \Rightarrow \underset{\sim}{k} \cdot \underset{\sim}{B_1} = k_0 B_x + k_0 B_z = k_0\left(\cancel{-B_0 \frac{k_0}{\omega} U_0} + \cancel{B_0 \frac{k_0}{\omega} U_0}\right) = 0$ ✓

7. Thus, for the fast/slow wave with $\underset{\sim}{k} = k_0 \hat{\underset{\sim}{x}} + k_0 \hat{\underset{\sim}{z}}$, we get

$$\boxed{\begin{aligned} \rho_1 &= \rho_0 \frac{\omega k_0}{\omega^2 - k_0^2 c_s^2} U_0 \\ p_1 &= \gamma p_0 \frac{\omega k_0}{\omega^2 - k_0^2 c_s^2} U_0 \\ U_x &= U_0 \\ U_y &= 0 \\ U_z &= \frac{k_0^2 c_s^2}{\omega^2 - k_0^2 c_s^2} U_0 \\ B_x &= -B_0 \frac{k_0}{\omega} U_0 \\ B_z &= B_0 \frac{k_0}{\omega} U_0 \end{aligned}}$$

8. Let's look at the ~~total~~ pressure ~~term~~ for fast/slow wave in the simple case $c_s^2 = v_A^2$.

V. B. (Continued)

a. $-\frac{1}{\rho_0}\nabla\left(p+\frac{B^2}{2\mu_0}\right) \Rightarrow \underset{\sim}{k}\left(\frac{p_1}{\rho_0}+\frac{\underset{\sim}{B_0}\cdot\underset{\sim}{B_1}}{\mu_0\rho_0}\right)$

b. Since $\underset{\sim}{k}=k_0\hat{\underset{\sim}{x}}+k_0\hat{\underset{\sim}{z}}$, both components of pressure force are same.

$$= k_0\left[\frac{\gamma p_0}{\rho_0}\frac{\omega^2}{\omega^2-k_0^2c_s^2}\frac{k_0}{\omega}U_0+\frac{B_0\left(B_0\frac{k}{\omega}U_0\right)}{\mu_0\rho_0}\right]=k_0\left[c_s^2\frac{\omega^2}{\omega^2-k_0^2c_s^2}+v_A^2\right]\frac{k_0U_0}{\omega}$$

c. For $c_s^2=v_A^2$, $\omega^2=\frac{k_0^2}{\cancel{2}}(\cancel{2}c_s^2)\left[1\pm\sqrt{1-\frac{2c_s^4}{(2c_s^2)^2}}\right]=k_0^2c_s^2\left(1\pm\sqrt{\tfrac{1}{2}}\right)$

d. $$=\frac{k_0^2c_s^2}{\omega}U_0\left[\frac{\omega^2}{\omega^2-k_0^2c_s^2}+1\right]$$

e. NOTE: $\omega^2-k_0c_s^2=k_0^2c_s^2\left[\left(1\pm\sqrt{\tfrac{1}{2}}\right)-1\right]=\pm k_0^2c_s^2\sqrt{\tfrac{1}{2}}$ (upper sign: Fast; lower sign: Slow)

f. Thus, $$=\frac{k_0^2c_s^2U_0}{\omega}\left[\frac{\cancel{k_0^2c_s^2}\left(1\pm\sqrt{\frac{1}{2}}\right)}{\pm\cancel{k_0^2c_s^2}\sqrt{\frac{1}{2}}}+1\right]=\frac{\cancel{k_0^2c_s^2}}{\cancel{k_0^2c_s^2}\left(1\pm\sqrt{\frac{1}{2}}\right)}\omega U_0\left[\pm\left(\sqrt{2}\pm 1\right)+1\right]$$

$$=\frac{\omega U_0}{\left(1\pm\sqrt{\frac{1}{2}}\right)}\left(2\pm\sqrt{2}\right)=\frac{\omega U_0\, 2\,\cancel{(1\pm\sqrt{2})}}{\cancel{1\pm\sqrt{2}}}=2\omega U_0=2k_0c_s\sqrt{1\pm\left(\tfrac{1}{2}\right)^{\frac{1}{2}}}\,U_0$$

g. Thus, the pressure force is

$$=\begin{cases}\left(4+2\sqrt{2}\right)^{\frac{1}{2}}k_0c_sU_0 & \text{Fast} \quad \omega=k_0c_s\sqrt{1+\sqrt{\tfrac{1}{2}}}\\ \left(4-2\sqrt{2}\right)^{\frac{1}{2}}k_0c_sU_0 & \text{Slow} \quad \omega=k_0c_s\sqrt{1-\tfrac{1}{\sqrt{2}}}\end{cases}$$