Lecture 21 Force-Balanced MHD Equilibria

I. Review of MHD Equilibria

A. Momentum Eqn:

\[ \rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + j \times B - \rho \nabla \phi \]

Equilibrium has zero net force.

2. Neglecting gravity (generally valid for laboratory plasmas)

\[ \nabla p = j \times B \] required for MHD Equilibrium

a. If \( j \times B = 0 \) \((\nabla p \ll j \times B)\) Force-Free

b. If \( j \times B \neq 0 \) Force-Balanced

3. Hopf's Theorem:

a. A torus is simplest topological surface satisfying \( \nabla B = 0 \) \& \( B \nabla \times B = 0 \)

Thus, to obtain a torus plasma, magnetic field lines lie on closed surfaces of constant pressure!

Negligible Toroidal Surfaces of constant pressure.

4. Force-Free Equilibrium Solutions

a. Flux Ropes

b. Reverse Field Pinch (RFP)

II. Force-Balanced Equilibria:

A. Using Ampere's Law, we may write force balance as

\[ \nabla \left( \rho \left( \frac{\mathbf{B}^2}{2\mu_0} \right) \right) = \frac{(\mathbf{B} \cdot \nabla) \mathbf{B}}{\mu_0} \]

\( \Rightarrow \text{Let's focus on cylindrical and toroidal geometries} \)
III. Force-Balanced Equilibria in Cylindrical Geometries

A. General Case:

1. Well focus on the family of solutions with a magnetic field with no radial component and only radial dependence,

\[ B = B_\phi(r) \hat{\phi} + B_z(r) \hat{z} \quad (B_r = 0) \]

2. The radial component of the force balance (using NRL p. 6-7) gives

\[ \frac{d}{dr} \left( p(r) + \frac{[B_\phi(r)]^2}{2\mu_0} + \frac{[B_z(r)]^2}{2\mu_0} \right) = -\frac{\frac{B_\phi(r)^2}{\mu_0 r}}{\mu_0 r} \]

NOTE: Magnetic tension term depends only on \( B_\phi \).

3. Ampère's Law

\[ j = \frac{1}{2\pi} \mathbf{\nabla} \times \mathbf{B} \]

4. Force balance depends on 3-functions

\[ p(r), \quad B_\phi(r), \quad B_z(r) \]

a. If two are given, the third is determined by force balance (solution to differential equation)

b. We still have an infinite number of possible solutions

⇒ Certain limiting cases simplify the force balance equation.

B. The Z-Pinch

1. Here, we take \( B_z = 0 \). Therefore \( j_\phi = 0 \).

2. \[ \frac{d}{dr} \left( p + \frac{B_\phi^2}{2\mu_0} \right) = -\frac{B_\phi^2}{\mu_0 r} \]

a. The radial pressure gradient is entirely confined by \( B_\phi \).

b. \( B_\phi \) is entirely generated by an axial current \( j_z \).
3. Consider a plasma of radius \( a \) with a constant axial current density inside the plasma, \( j_z = j_0 \).

a. Total Current: 
\[
I_0 = \int_0^a 2\pi r \, dr \, j_0 = \pi a^2 j_0 \implies j_0 = \frac{I_0}{\pi a^2}
\]

b. Calculate the resulting \( B\phi \) from Ampere's Law:

\[
j_z = \frac{1}{\mu_0} \frac{df}{dr} (r \cdot B\phi)
\]

i) \( \int_0^a \frac{df}{dr} (r \cdot B\phi) = rB\phi = \int_0^r \frac{df}{dr} (r \cdot \mu_0 \mu \, j_0) \, dr \) where \( j_0 = \left\{ \begin{array}{ll}
\frac{j_0}{2\pi a^2} & r \leq a \\
0 & r > a
\end{array} \right. \)

\[
= \left\{ \begin{array}{ll}
\frac{\mu_0 I_0}{2\pi a^2} r & r \leq a \\
\frac{\mu_0 I_0}{2\pi a^2} & r > a
\end{array} \right.
\]

iii) \( B\phi(r) = \left\{ \begin{array}{ll}
\frac{\mu_0 I_0}{2\pi a^2} r & r \leq a \\
\frac{\mu_0 I_0}{2\pi a} & r > a
\end{array} \right. \)

\[
c. \text{ Calculate the pressure } p(r) \text{.}
\]

i) \( \frac{dp}{dr} = -\frac{\partial}{\partial r} \left( \frac{B\phi^2}{2\mu_0} \right) = \frac{B\phi^2}{2\mu_0} \left( \begin{array}{ll}
-\frac{2\mu_0 I_0^2 r}{4\pi^2 a^4} & r = a \\
0 & r > a
\end{array} \right) \)

ii) At the edge of the plasma, \( r = a \), we take \( p(a) = 0 \).
iii) Thus \[
\int_0^a \frac{dP}{dr} dr = P(a) - P(r) = \int_r^a \frac{2 \mu_0 \mu_0^2 r}{4 \pi^2 a^4} dr = -\frac{\mu_0 \mu_0^2}{4 \pi^2 a^2} (1 - \frac{r^2}{a^2})
\]

iv) Finally \[
P(r) = \frac{4 \mu_0^2}{(2 \pi a)^2} \left(1 - \frac{r^2}{a^2}\right)
\]

r ≤ a

5. Profiles of p & B_p

Magnetic pressure and magnetic tension contribute thermal pressure.

6. The Z-Pinch at Sandia National Laboratories

I = 70 MA

Frozen D₂ (deuterium) fibres

a. Enormous magnetic pressure (and tension) due to axial current confining plasma at deuterium in small volume
b. Unstable to “Sausage” instability

\( I \) produces optically x-rays from hot plasma, useful for studies of high-energy-density plasmas!

(MHD Stability)

(Next Semester we'll look at)
Lecture 21 (Continued)

II. Continued

C. The "Thea" Pinch

1. In this case, we take \( \mathbf{B}_\phi = 0 \) and can solve for \( p(r) \) in terms of \( B_z(r) \).

\[ \text{This is a homework problem!} \]

Current in \( \phi \) direction produces \( B_z \) that confines pressure.

III. Force-Balanced Equilibria in Toroidal Geometries

A. 1. Although the cylindrical cases give us a good intuition of force-balanced MHD Equilibrium, it is toroidal geometries that are necessary to have confined in 3-D (Kaufman theorem tells us we must at least have a torus).

2. We now consider toroidal geometries with symmetry in the toroidal direction \( \phi \).

B. Magnetic Flux Coordinates:

1. For \( \frac{\partial B_z}{\partial \phi} = 0 \), \( \nabla \cdot \mathbf{B} = 0 \) implies \( \frac{1}{r} \frac{\partial}{\partial r} (r B_r) + \frac{\partial B_z}{\partial z} = 0 \).

2. If we use the vector potential \( \mathbf{B} = \nabla \times \mathbf{A} \), then \( \nabla \cdot \mathbf{B} = 0 \) is automatically satisfied.
3. The toroidal symmetry implies a simplification ($\frac{\partial}{\partial r} = 0$)

$$B_r = (\nabla \times A)_r = - \frac{1}{r} \frac{\partial A_z}{\partial z} = - \frac{\partial A\phi}{\partial z}$$

$$B_z = (\nabla \times A)_z = \frac{1}{r} \frac{\partial}{\partial r} (r A\phi) - \frac{1}{r} \frac{\partial A\phi}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} (r A\phi)$$

- Both $B_r$ & $B_z$ depend only on $A\phi$.

4. Defining a flux function $A(r, z) = r A\phi$, we get

$$B_r = -\frac{1}{r} \frac{\partial A}{\partial z}$$

$$B_z = \frac{1}{r} \frac{\partial A}{\partial r}$$

5. **Note:** For $A(r, z)$,

$$B \cdot \nabla A = B_{ rz} A_{ rz} + B_{ rz} A_{ rz} = B_r (B_z) + B_z (B_r) = 0$$

- Thus, $\frac{\partial A}{\partial z} = 0$ over $B = 0$.

- Magnetic field line must lie on surfaces of $A$-constant.

- But, we also know $B \cdot \nabla p = 0$ from the force balance, so $p = p(A)$.

6. Consider the $\phi$-component of $\nabla \times B = \nabla p$:

$$J_r B_r - J_r B_z = -\frac{1}{r} \frac{\partial p}{\partial \phi}$$

but $-\frac{1}{r} \frac{\partial p}{\partial \phi} = 0$ since $A = A(r, z)$.

**NB:**

b. Ampère's law gives:

$$J_r = -\frac{1}{\mu_0} (\nabla \times B)_r = \frac{1}{\mu_0} \left[ \frac{\partial A_z}{\partial z} - \frac{\partial A\phi}{\partial z} \right]$$

$$J_z = -\frac{1}{\mu_0} (\nabla \times B)_z = \frac{1}{\mu_0} \left[ \frac{\partial A_r}{\partial r} - \frac{\partial A\phi}{\partial r} \right]$$

As with $B$, in terms of $A$, $J$ depends only on $B \phi (r, z)$.

c. Analogous, redefine a second flux function $E = r \Phi \phi$ to satisfy $J_r B_r - J_r B_z = 0$. 
Lecture #21 (Continued)

III. B & G (Continued)

d. It follows that \( B_z \frac{\partial F}{\partial z} + B_z \frac{\partial F}{\partial z} = 0 \Rightarrow B \cdot \nabla F = 0 \)

and so \( F = F(\gamma) \)

7. Thus, we can express the toroidal symmetric magnetic field in terms of two scalar flux functions \( \Psi(r,z), \Phi(r,z) \):

\[
B = \left( -\frac{1}{r} \frac{\partial \Psi}{\partial z} \right) \hat{\hat{z}} + \left( \frac{1}{r} \frac{\partial \Phi}{\partial r} \right) \hat{\hat{r}} + \left( \frac{F(\gamma)}{r} \right) \hat{\hat{\phi}}
\]

Magnetic Field expressed using Magnetic Flux Coordinates

b. This can also be expressed \( B = \nabla \Phi \times \hat{\hat{z}} + \frac{F(\gamma)}{r} \hat{\hat{\phi}} \)

C. The Grad-Shapiro Equation:

1. Now, let's consider the \( r \)-component of the force balance.

\[
J \cdot B_z = J_z B_r = \frac{\partial \Phi}{\partial r}
\]

2. First use Ampere's Law to calculate \( J \cdot B \)

\[
\nabla \times \Phi = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} \Rightarrow \nabla \times \Phi = -\frac{J_z}{r} \frac{\partial \Phi}{\partial z} - \frac{2}{r^2} \frac{F(\gamma)}{r} \Phi = -\frac{1}{r} \Delta \Phi
\]

\[
B_r = J \times \Phi
\]

\[
B_z = J \times \Phi
\]

3. Similarly

\[
\nabla \times \Phi = \frac{1}{r} \frac{\partial \Phi}{\partial r}(r B_r) - \frac{1}{r^2} \frac{\partial \Phi}{\partial r} \Rightarrow \frac{1}{r} \frac{\partial F(\Phi)}{\partial r} = \frac{1}{r \Phi} \frac{\partial F}{\partial \Phi}
\]

\[
F = r B_r
\]

4. Thus \( -\frac{1}{r} \Delta \Phi B_r - \frac{J_z}{r} \frac{\partial F_{\Phi}}{\partial r} = \mu_o \frac{\partial \Phi}{\partial r} \)

b. Again, substituting for \( B_r \) & \( B_z \) gives

\[
-\frac{1}{r} \Delta \Phi \left( \frac{1}{r} \frac{\partial \Phi}{\partial r} \right) = -\frac{1}{r} \frac{\partial F(\Phi)}{\partial r} = \mu_o \frac{\partial \Phi}{\partial r}
\]
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III. C. (Continued)

5. \( \delta F = \frac{dF^2}{dr} \) and \( p = p(\Psi) \), so

\[
\frac{\partial \Psi}{\partial r} = \frac{\delta F}{\delta r} \quad \text{and} \quad \frac{\partial p}{\partial r} = \frac{\partial}{\partial r} \left( \frac{\partial \Psi}{\partial r} \right).
\]

Thus,

\[
\frac{1}{r^2} \Delta \Phi \left( \frac{\partial \Psi}{\partial r} \right) = -\frac{1}{2 \rho} \frac{\partial}{\partial \rho} \left( \frac{\partial \Phi \Phi}{\partial \rho} \right) = \text{Grad-Shafranov Equation}
\]

6. Finally, we obtain

\[
\Delta^* \Psi = -\lambda \rho \frac{\partial^2 H}{\partial \rho^2} - \frac{1}{2} \frac{\partial F^2}{\partial \Psi}
\]

where \( \Delta^* \Psi = \rho \frac{\partial}{\partial \rho} \left( \frac{\partial \Psi}{\partial \rho} \right) + \frac{\partial^2 \Psi}{\partial z^2} \)

D. Application

1. Grad-Shafranov equation is used to calculate

\[ \text{Magnetostatic Equilibria in axisymmetric toroidal systems.} \]

2. In practice:
   a. Specify \( p(\Psi) \) and \( F(\Psi) \)

   Pressure \hspace{1cm} \text{Toroidal Field Function}

   b. Solve Grad-Shafranov Eq. (Numerically) with specified boundary conditions for \( \Psi(r, z) \)

   c. Pressure profile is then determined \( p = p(\Psi(r, z)) \)

3. Exact analytical solutions, known as Solov'ev equilibria, are often used in analysis of toroidal magnetic fusion devices.