Lecture #24 Cold vs. Warm Unmagnetized Plasma Waves

I. Longitudinal Modes in Warm vs. Cold Unmagnetized Plasmas

A. Longitudinal Modes

1. Last time, we found that, in a cold plasma, Longitudinal modes, or Plasma Oscillations, do not propagate.

2. If a plasma has a finite temperature — typically the case in laboratory, space, and astrophysical plasmas — Longitudinal modes *do* propagate as
   Langmuir Waves or Ion acoustic Waves

3. To demonstrate this, we will calculate the dispersion relation when the electron temperature is finite $T_e \neq 0$.
   a. We still take $T_i = 0$ (or $T_i \ll T_e$), a *cold ion approximation* to keep the mathematics simple.
   b. In laboratory plasmas, it is common to have $T_i < T_e$.

ASIDE: *Why?*

1) To ionize a laboratory plasmas, often a hot cathode is used.
2) The hot cathode emits very fast electrons which stream along the magnetic field line leading to ionization of the plasma (collisional ionization)
3) From *Lecture #1*, we learned about the timescale to achieve thermal equilibrium between species via collisions.
   a. Plasma electrons can easily gain energy from (hot electron)–(electron) collisions on a timescale $\tau_{ee} \sim \frac{1}{\nu_{ee}}$
   b. But, for ions to reach thermal equilibrium with hot electrons requires a time $\tau_{ie} \sim \frac{m_i}{m_e} \tau_{ee} \sim 1836\, \tau_{ee} \gg \tau_{ee}$!

II. Dispersion Relation for a Warm, Unmagnetized Plasma with $T_i \ll T_e$

A. Equations:

1. We now may not neglect the electron pressure term in the electron momentum equation, but we'll continue to neglect ion pressure

Continuity: $\frac{\partial n_i}{\partial t} + \underset{\sim}{U}_i \cdot \nabla n_i = -n_i \nabla \cdot \underset{\sim}{U}_i$ $\qquad$ $\frac{\partial n_e}{\partial t} + \underset{\sim}{U}_e \cdot \nabla n_e = -n_e \nabla \cdot \underset{\sim}{U}_e$

Momentum: $m_i n_i \left[\frac{\partial \underset{\sim}{U}_i}{\partial t} + \underset{\sim}{U}_i \cdot \nabla \underset{\sim}{U}_i\right] = q_i n_i (\underset{\sim}{E} + \underset{\sim}{U}_i \times \underset{\sim}{B})$

$$m_e n_e \left[\frac{\partial \underset{\sim}{U}_e}{\partial t} + \underset{\sim}{U}_e \cdot \nabla \underset{\sim}{U}_e\right] = -\nabla p_e + q_e n_e (\underset{\sim}{E} + \underset{\sim}{U}_e \times \underset{\sim}{B})$$

Maxwell's Eqs: $\nabla \cdot \underset{\sim}{E} = \frac{\rho_c}{\epsilon_0}$ $\qquad$ $\nabla \times \underset{\sim}{E} = \frac{\partial B}{\partial t}$

$\nabla \cdot \underset{\sim}{B} = 0$ $\qquad$ $\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{j} + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$

2. Closure: We need an additional equation to relate p_e to the other variables to close this system.

3. Isothermal Equation of State for electrons

a. $\frac{d}{dt}\left(\frac{p_e}{n_e^\gamma}\right) = 0$

b. This time we take $\gamma = 1$, corresponding to an isothermal eq. of state.

c. $\Rightarrow p_e = n_e \underset{\text{constant}}{\underset{\uparrow}{C}}$ $\qquad$ We take the constant $C = k \underset{\uparrow}{T_e} = \text{const}$

Thus T_e is constant.

$$\Rightarrow \boxed{p_e = n_e k T_e}$$

d. Because electrons can move very rapidly in any direction in an unmagnetized plasma, for slow motions (low freq) an isothermal eq. of state is often used.

Lecture #24 (Continued)

II. (Continued)

B. Linearization and Fourier Transform of Equations

1. This has been done in detail in Lecture #22, so we'll only treat the new electron momentum equation.

2. Assume:

$$n_e = n_{e0} + \epsilon n_{e1}$$
$$\underset{\sim}{U}_e = \qquad \epsilon \underset{\sim}{U}_{e1}$$
$$p_e = p_{e0} + \epsilon p_{e1}$$
$$\underset{\sim}{E} = \qquad \epsilon \underset{\sim}{E}_1$$
$$\underset{\sim}{B} = \qquad \epsilon \underset{\sim}{B}_1$$

3. Linearize:

$$\epsilon m_e n_{e0} \frac{\partial \underset{\sim}{U}_{e1}}{\partial t} + \epsilon^2 m_e n_{e1} \frac{\partial U_{e1}}{\partial t} + \epsilon^2 \overset{m_e n_{e0}}{\underset{\sim}{U}_{e1}} \cdot \nabla \underset{\sim}{U}_e + \epsilon^3 m_e n_{e1} \underset{\sim}{U}_{e1} \cdot \nabla \underset{\sim}{U}_{e1}$$

$$= -\cancelto{0}{\nabla p_{e0}} + \epsilon \nabla p_{e1} + \epsilon q_e n_{e0} \underset{\sim}{E}_1 + \epsilon^2 q_e n_{e1} \underset{\sim}{E}_1 + \epsilon^2 q_e n_{e0} \underset{\sim}{U}_{e1} \times \underset{\sim}{B}_1 + \epsilon^2 q_e n_{e1} \underset{\sim}{U}_{e1} \times \underset{\sim}{B}_1$$

$$O(\epsilon) \Rightarrow \boxed{m_e n_{e0} \frac{\partial \underset{\sim}{U}_{e1}}{\partial t} = -\nabla p_{e1} + q_e n_{e0} \underset{\sim}{E}_1}$$

4. **NOTE:** Equation of State: $p_{e0} + \epsilon p_{e1} = n_{e0} k T_e + \epsilon n_{e1} k T_e$

   a. In our equilibrium, we assume $p_{e0} = n_{e0} k T_e$, so we are left with

$$p_{e1} = n_{e1} k T_e \qquad \text{(Remember } T_e = \text{const) Isothermal}$$

5. Fourier Transform: ~~All~~ Plane wave solutions $\propto e^{i(\underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t)}$

$$\frac{\partial}{\partial t} \Rightarrow -i\omega \qquad \nabla \Rightarrow i\underset{\sim}{k}$$

   a. $-i m_e n_{e0} \omega \underset{\sim}{U}_{e1} = -i\underset{\sim}{k} p_{e1} + q_e n_{e0} \underset{\sim}{E}_1$

   b. $$\boxed{\omega \underset{\sim}{U}_{e1} = \underset{\sim}{k} \frac{p_{e1}}{m_e n_{e0}} + i \frac{q_e}{m_e} \underset{\sim}{E}_1}$$

Lecture #24 (Continued)

II. (Continued)

C. System of Equations:

① $\omega n_{i_1} = n_{i_0} \underset{\sim}{k}\cdot\underset{\sim}{U}_{i1}$ ② $\omega n_{e1} = n_{e0}\underset{\sim}{k}\cdot\underset{\sim}{U}_{e1}$

③ $\omega \underset{\sim}{U}_{i1} = i\frac{q_i}{m_i}\underset{\sim}{E}_1$ ④ $\omega \underset{\sim}{U}_{e1} = \underset{\sim}{k}\frac{p_{e1}}{m_e n_{e0}} + i\frac{q_e}{m_e}\underset{\sim}{E}_1$

⑤ $i\underset{\sim}{k}\cdot\underset{\sim}{E}_1 = \frac{1}{\epsilon_0}(q_i n_{i_1} + q_e n_{e1})$ ⑥ $\omega \underset{\sim}{B}_1 = \underset{\sim}{k}\times\underset{\sim}{E}_1$

⑦ $\omega \underset{\sim}{E}_1 = \frac{-i}{\epsilon_0}\left(q_i n_{i0}\underset{\sim}{U}_{i1} + q_e n_{e0}\underset{\sim}{U}_{e1}\right) - c^2\underset{\sim}{k}\times\underset{\sim}{B}_1$ ⑧ $p_{e1} = n_{e1} kT_e$

D. Solve in terms of $\underset{\sim}{E}_1$ only to yield matrix equation

1. Using ①, ③, ⑤, and ⑧, we can solve ④ in terms of $\underset{\sim}{U}_{e1}$ and $\underset{\sim}{E}_1$ only. The result is

$$\boxed{\omega\underset{\sim}{U}_{e1} = \frac{iq_e}{m_e}\left[\frac{C_e^2}{\omega_{pe}^2}\left(1-\frac{\omega_{pi}^2}{\omega^2}\right)\underset{\sim}{k}(\underset{\sim}{k}\cdot\underset{\sim}{E}_1) + \underset{\sim}{E}_1\right]} \quad ⑨$$

where we have used the definition $\boxed{C_e^2 \equiv \frac{kT_e}{m_e}}$

and definitions for the plasma frequency $\omega_{ps}^2 \equiv \frac{n_{s_0} q_s^2}{\epsilon_0 m_s}$

2. Using ③, ⑥, and ⑨, we can solve ⑦ solely in terms of E_1, obtaining

$$\left[\omega^2 - \omega_p^2 - c^2k^2\right]\underset{\sim}{E}_1 - \left[C_e^2\left(1-\frac{\omega_{pi}^2}{\omega^2}\right) + c^2\right]\underset{\sim}{k}(\underset{\sim}{k}\cdot\underset{\sim}{E}_1) = 0$$

where $\omega_p^2 = \omega_{pi}^2 + \omega_{pe}^2$

3. **NOTE:** $\underset{\sim}{E}_1 = E_{T_1}\hat{\underset{\sim}{e}}_1 + E_{T_2}\hat{\underset{\sim}{e}}_2 + E_L\hat{\underset{\sim}{k}}$

and $\underset{\sim}{k}(\underset{\sim}{k}\cdot\underset{\sim}{E}_1) = k^2\,\hat{\underset{\sim}{k}}(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{E}_1) = k^2 E_L\hat{\underset{\sim}{k}}$

Lecture #24 (Continued)

II.D. (Continued)

4. The three equations are then:

$\hat{\underset{\sim}{e}}_1$: $(\omega^2-\omega_p^2-c^2k^2)E_{T_1}=0$

$\hat{\underset{\sim}{e}}_2$ $(\omega^2-\omega_p^2-c^2k^2)E_{T_2}=0$

$\hat{\underset{\sim}{k}}$ $(\omega^2-\omega_p^2-\cancel{c^2k^2}-c_e^2k^2(1-\frac{\omega_{pi}^2}{\omega^2})+\cancel{c^2k^2})E_L=0$

5. Thus, our Matrix equation is

$$\begin{pmatrix} \omega^2-\omega_p^2-c^2k^2 & 0 & 0 \\ 0 & \omega^2-\omega_p^2-c^2k^2 & 0 \\ 0 & 0 & \omega^2-\omega_p^2-k^2c_e^2(1-\frac{\omega_{pi}^2}{\omega^2}) \end{pmatrix}\begin{pmatrix} E_{T_1} \\ E_{T_2} \\ E_L \end{pmatrix}=0$$

a. NOTE! The only change from the Cold Plasma Equations is in the Longitudinal term.
⇒ The inclusion of electron pressure modifies the plasma oscillation.

b. The transverse Modified Light Waves are unaffected by the pressure gradient term. This is because $\underset{\sim}{k}\cdot\underset{\sim}{E}_1=0$ ~~zero~~ for these modes, so there are no density fluctuations and thus no pressure fluctuations for the transverse mode.

6. Longitudinal Term: $(E_L\neq 0,\ \underset{\sim}{E}_T=0)$

$$\boxed{\omega^2-\omega_p^2-k^2c_e^2\left(1-\frac{\omega_{pi}^2}{\omega^2}\right)=0}$$

a. General Solution:

$$\boxed{\omega^2=\frac{(\omega_p^2+k^2c_e^2)}{2}\left(1\pm\sqrt{1-\frac{4k^2c_e^2\omega_{pi}^2}{(\omega_p^2+k^2c_e^2)^2}}\right)}$$

b. There are four seperate roots, corresponding to two wave modes:

1. Langmuir Waves
2. Ion Acoustic Waves

II. D (Continued)

6. (Continued)

c. In general, the solution is difficult to interpret.

$\Rightarrow$ Plasma physicists often take <u>limits</u> of the solution.

$$\boxed{\text{Finding simplified ~~limit~~ solutions in limiting cases is an important skill for interpreting physical behavior.}}$$

d. For Example: $\omega^2 - \underset{\omega_p^2 \simeq \omega_{pe}^2}{\underset{\uparrow}{\omega_{pe}^2}} - k^2 C_e^2\left(1 - \frac{\omega_{pi}^2}{\omega^2}\right) = 0$

1. When $\omega > \omega_{pe}$ (and $\omega_{pe} \gg \omega_{pi}$), $\frac{\omega_{pi}^2}{\omega^2} \ll 1$, so

$$\boxed{\omega^2 - \omega_{pe}^2 - k^2 C_e^2 = 0}\quad \text{High Frequency Limit}$$

2. When $\omega < \omega_{pi}$ (and $\omega_{pe} \gg \omega_{pi}$), $\omega^2 \ll \omega_{pe}^2$, so

$$\boxed{-\omega_{pe}^2 - k^2 C_e^2\left(1 - \frac{\omega_{pi}^2}{\omega^2}\right) = 0}\quad \text{Low Frequency Limit}$$

E. <u>Langmuir Waves</u> High Frequency, $\omega > \omega_{pe}$

1. $\boxed{\omega^2 = \omega_{pe}^2 + k^2 C_e^2}$

2. <u>NOTE:</u> $\omega_{pe}^2 \lambda_{De}^2 = \left(\frac{\cancel{n_{0e} q_e^2}}{\cancel{\epsilon_0} m_e}\right)\left(\frac{\cancel{\epsilon_0} k T_e}{\cancel{n_{0e} q_e^2}}\right) = \frac{k T_e}{m_e} = C_e^2$

Thus

$$\omega^2 = \omega_{pe}^2 (1 + k^2 \lambda_{De}^2)$$

3. <u>Wavelength limits:</u>

a. <u>Long wavelength:</u> $k^2 \lambda_{De}^2 \ll 1$ $\Rightarrow \frac{(2\pi)^2 \lambda_{De}^2}{\lambda^2} \ll 1$ or $\lambda \gg \lambda_{De}$

(Non-propagating) $\boxed{\omega^2 = \omega_{pe}^2}$ Usual electron plasma oscillations as in cold plasma theory

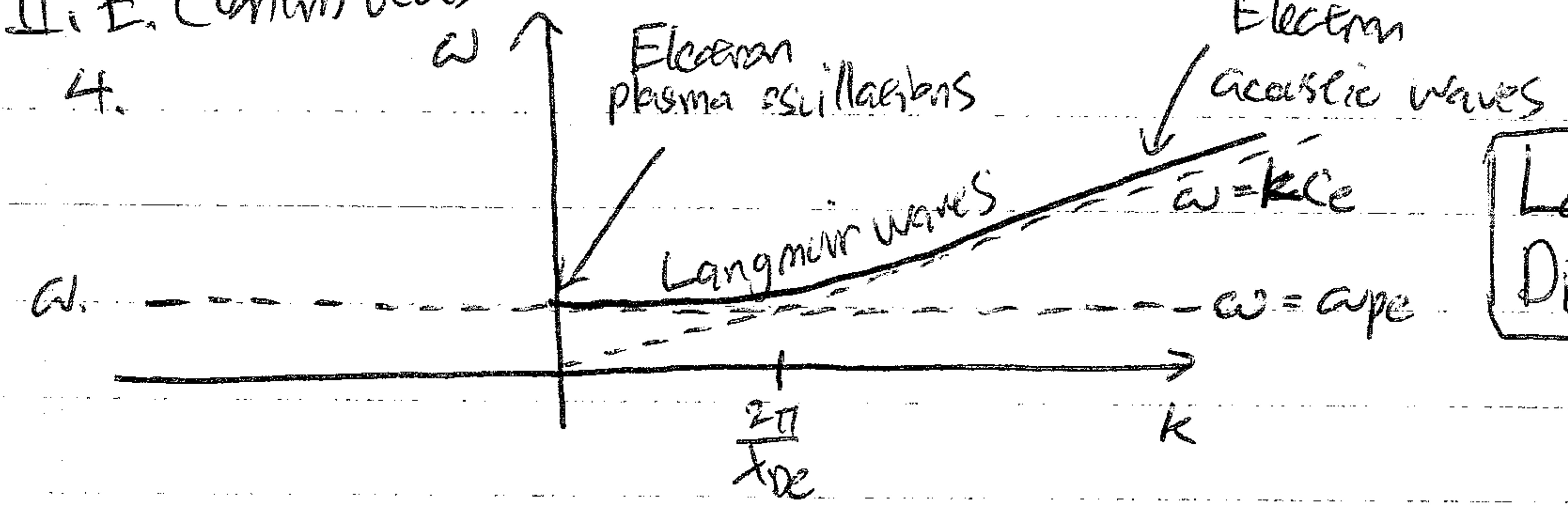
b. <u>Short wavelength:</u> $k^2 \lambda_{De}^2 \gg 1$ (or $\lambda \ll \lambda_{De}$)

$\boxed{\omega^2 = k^2 C_e^2}$ Electron acoustic wave (Non-dispersive)

Lecture #24 (Continued)

II. E. (Continued)

4.



$\boxed{\text{Langmuir Wave Dispersion Relation}}$

5. <u>IMPORTANT NOTE</u>: a. At wavelengths such that $k\lambda_{De} \gtrsim 1$, the phase velocity of the waves $V_p = \frac{\omega}{k} \simeq c_e$ is very near the electron thermal velocity $V_{te} = \sqrt{\frac{2kT_e}{m_e}}$.

b. The waves are resonant with the electrons. This leads to strong collisionless damping of the electron acoustic wave called <u>Landau damping</u>.

c. Thus, electron acoustic waves don't really occur.

d. This is a failure of the two-fluid theory.

⇒ Kinetic theory is required to ~~[illegible]~~ describe collisionless damping.

F. <u>Ion Acoustic Waves</u>: Low Frequency, $\omega < \omega_{pi}$

1. $-\omega_{pe}^2 - k^2 c_e^2 \left(1 - \frac{\omega_{pi}^2}{\omega^2}\right) = 0 \Rightarrow \omega^2 = \frac{\omega_{pi}^2 k^2 c_e^2}{\omega_{pe}^2 + k^2 c_e^2} = \frac{\omega_{pi}^2}{\omega_{pe}^2} \frac{k^2 c_e^2}{(1 + k^2 \lambda_{De}^2)}$

a. NOTE: $\frac{\omega_{pi}^2}{\omega_{pe}^2} = \frac{\left(\frac{\cancel{n_i q_i^2}}{\cancel{\epsilon_0} m_i}\right)}{\left(\frac{\cancel{n_e q_e^2}}{\cancel{\epsilon_0} m_e}\right)} = \frac{m_e}{m_i}$ $\Rightarrow \omega^2 = \frac{k^2 \left(\frac{kT_e}{\cancel{m_e}}\right) \frac{\cancel{m_e}}{m_i}}{1 + k^2 \lambda_{De}^2}$

b. Ion Acoustic Dispersion Relation $\boxed{\omega^2 = \frac{k^2 c_i^2}{1 + k^2 \lambda_{De}^2}}$

c. <u>DEF</u>: Ion Acoustic Speed $\boxed{c_i^2 = \frac{kT_e}{m_i}}$ ← Electrons provide pressure ← Ions provide inertia.

Lecture #24 (Continued)

II. F. (Continued)

2. Wavelength Limits:

a. Long wavelength: $k^2 \lambda_{De}^2 \ll 1$

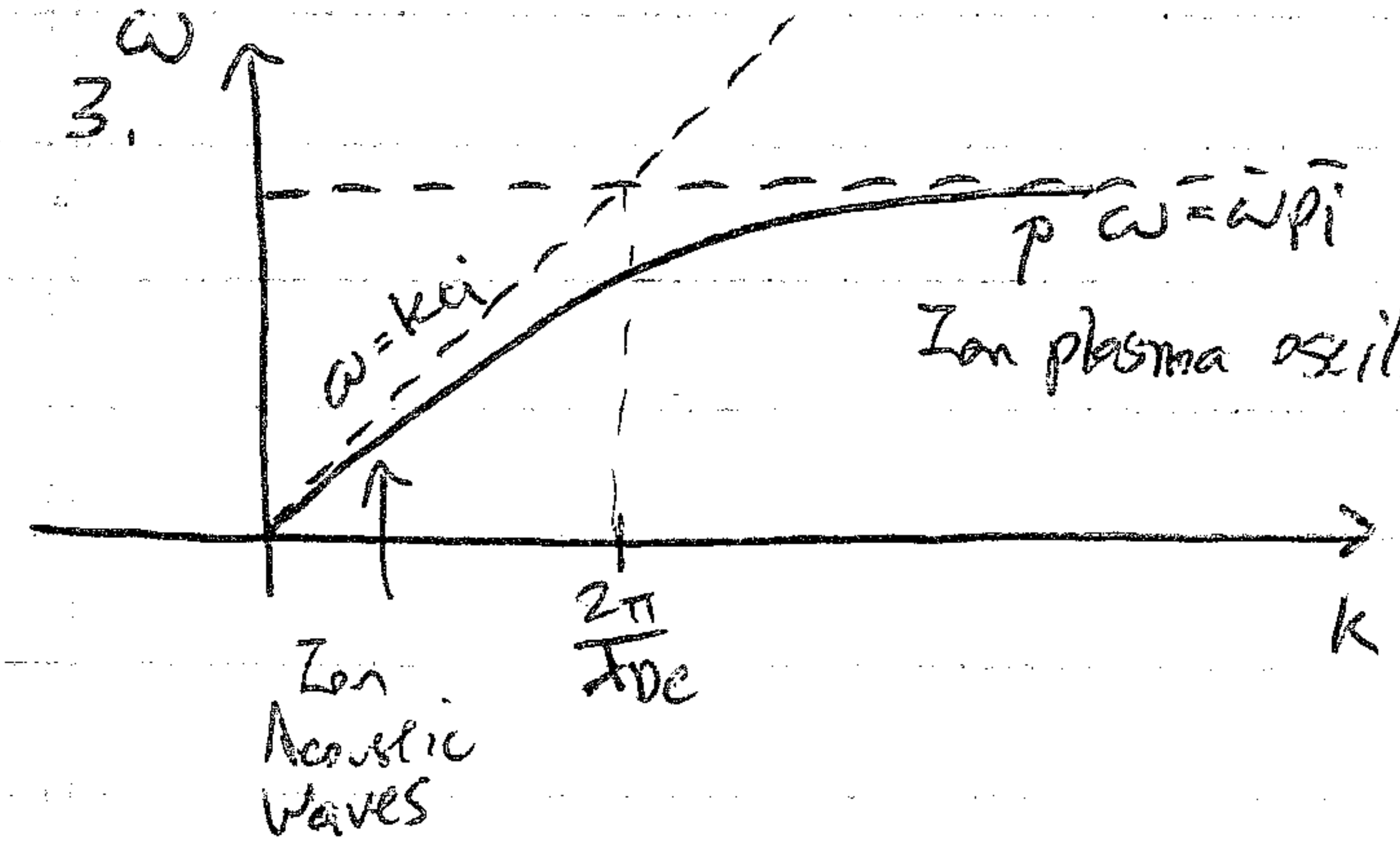
$\boxed{\omega^2 = k^2 c_i^2}$ Ion Acoustic Wave (Non-dispersive) $\frac{\omega}{k} = \cancel{\pm} c_i$

b. Short Wavelength: $k^2 \lambda_{De}^2 \gg 1$

1. $\omega^2 = \frac{\cancel{k^2} c_i^2}{\cancel{k^2} \lambda_{De}^2} = \frac{\left(\frac{\cancel{kT_e}}{m_i}\right)}{\left(\frac{\epsilon_0 \cancel{kT_e}}{n_{0e} q_e^2}\right)} = \frac{n_{0e} q_e^2}{\epsilon_0 m_i} = \frac{n_{0i} q_i^2}{\epsilon_0 m_i} = \omega_{pi}^2$

2. $\boxed{\omega^2 = \omega_{pi}^2}$ Plasma oscillations at ion plasma frequency.

3.



3a. If, $T_i = T_e$ (instead of $T_i \ll T_e$ as we assumed), then the ion acoustic phase speed $\frac{\omega}{k} = c_i = \sqrt{\frac{kT_e}{m_i}} \simeq \sqrt{\frac{2kT_i}{m_i}} = v_{t_i}$

b. Thus, when ion temperature is equal to electron temperature, ions thermal velocity is resonant with the wave phase speed.

c. Ion Acoustic waves experience strong collisionless Landau damping when $T_i = T_e$ (Again, kinetic description is necessary)

d. Only when $T_i \ll T_e$ do ion acoustic waves propagate with little damping.